MATH 525 – Final Exam

R. Hammack
May 12, 2016

Name: ____________________________

Directions. Answer the questions in the space provided. This is a closed-notes, closed book exam; no calculators, no computers and no formula sheets. You have three hours.

(1) A classroom has 2 rows of 8 seats each. There are 14 students, 5 of whom always sit in the first row and 4 of whom always sit in the back row. In how many ways can the students be seated?

(2) (a) Find the coefficient of $x^3y^2z^5$ in $(x+y+z)^{10}$.

(b) Find the number of integer solutions of $x+y+z+w=30$ that satisfy $x \geqslant 2$, $y \geqslant 0$, $z \geqslant -8$, $w \geqslant 5$.

(3) There are 20 sticks lined up in a row occupying 20 distinct positions:

||||||||||||||||||||,

and six of them are to be chosen. How many choices are there if there must be at least two sticks between any pair of chosen sticks?

(4) Use a combinatorial argument to prove that $\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$ for all positive integers a, b and n.

(5) Find and verify a formula for $\displaystyle\sum_{\substack{k,\ell,m \geqslant 0 \\ k+\ell+m=n}} \binom{a}{k}\binom{b}{\ell}\binom{c}{m}.$

[Suggestion: generalize your argument from question 4 on the previous page.]

(6) Find the number of integers between 1 and 10,000 (inclusive) which are *not* divisible by 4, 5, or 6.

(7) Consider the multiset $X = \{\infty \cdot a,\ \infty \cdot b,\ \infty \cdot c,\ \infty \cdot d\}$. Let h_n be the number of n-combinations of X that have no more than 4 a's, a multiple of 5 b's, at least 5 c's, and at least 2 d's.

(a) Find an ordinary generating function for h_n.

(b) Use your answer from part (a) to find a general formula for h_n.

(8) Let h_n be the number of n-digit numbers with all digits odd and no consecutive 3's. (By default, assume $h_0 = 1$.)

(a) Find a homogeneous recurrence relation for h_n.

(b) Solve the recurrence relation to find a general formula for h_n.

(9) Let h_n be the number of n-digit numbers with all digits odd and for which the digits 1 and 3 occur a *positive* even number of times.

(a) Find an exponential generating function for h_n.

(b) Use your answer from part (a) to find a general formula for h_n.

(10) The general term of a sequence $h_0, h_1, h_2, \ldots$ is a polynomial in n of degree 3. The first four entries of the 0th diagonal of its difference table are $1, -1, 3, 10$. Find the formula for h_n.