§ 8.4 A Geometric Problem

Today we answer the following question:

Given n hyperplanes in $\mathbb{R}^k$, how many regions do the divide $\mathbb{R}^k$ into?

Before delving into this questions let's review hyperplanes.

Recall: k-dimensional space is $\mathbb{R}^k = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \middle| \, x_i \in \mathbb{R} \right\}$

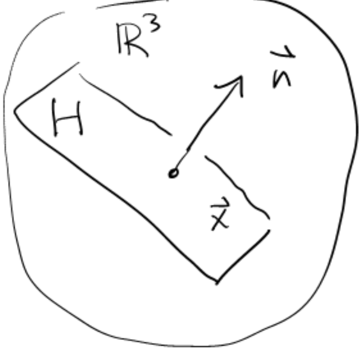
Notation $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \vec{x}$, $\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{bmatrix} = \vec{n}$, etc. We call these vectors.

A homogeneous hyperplane in $\mathbb{R}^k$ is $H = \{\vec{x} \in \mathbb{R}^k \mid \vec{n} \cdot \vec{x} = 0\}$ where $\vec{n} \in \mathbb{R}^k$ is a fixed non-zero vector.

H is null space of linear transformation $T: \mathbb{R}^k \to \mathbb{R}$ defined as $T(x) = n \cdot x$. By rank theorem $\dim(H) = \dim(\mathbb{R}^k) - \dim(\mathbb{R}) = k-1$.

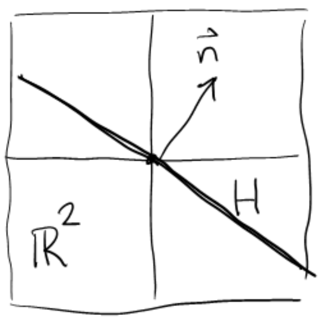
$\boxed{\text{Thus a hyperplane } H \text{ in } \mathbb{R}^k \text{ has dimension } k-1.}$

Example $k=3$



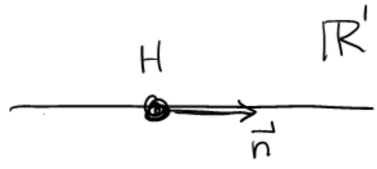
$H = \{\vec{x} \in \mathbb{R}^3 \mid \vec{n} \cdot x = 0\}$ is all vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{n}$. Thus H is a 2-D plane in $\mathbb{R}^3$

Example $k=2$



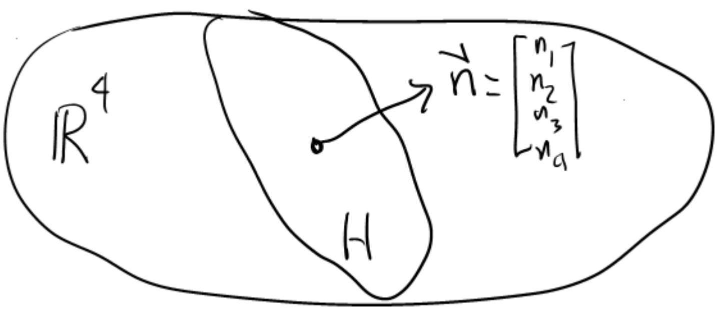
$H = \{\vec{x} \in \mathbb{R}^2 \mid \vec{n} \cdot x = 0\}$ is all vectors orthogonal to $\vec{n}$. Thus H is a 1-D line in $\mathbb{R}^2$

Example $k=1$



$H = \{\vec{x} \in \mathbb{R}^1 \mid \vec{n} \cdot x = 0\} = \{0\}$. Thus H is a 0-D point in $\mathbb{R}^1$

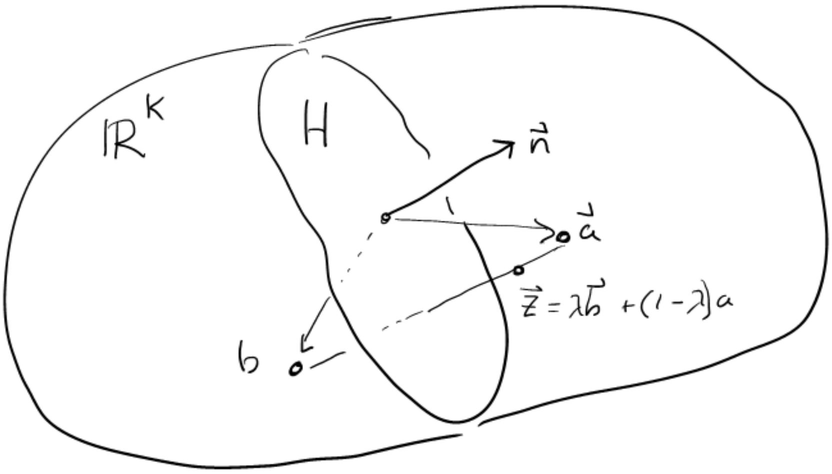
Example $k=4$ This is harder to visualize, but $H = \{\vec{x} \in \mathbb{R}^4 \mid \vec{n} \cdot \vec{x} = 0\}$ is a 3-D subspace of $\mathbb{R}^n$.



Basic Fact 1

A hyperplane $H = \{\vec{x} \in \mathbb{R}^k \mid \vec{n} \cdot \vec{x} = 0\}$ in $\mathbb{R}^k$ cuts $\mathbb{R}^k$ into two half-spaces

$$H^+ = \{ \vec{x} \in \mathbb{R}^k \mid \vec{n} \cdot \vec{x} > 0 \}$$
$$H^- = \{ x \in \mathbb{R}^k \mid \vec{n} \cdot \vec{x} < 0 \}$$



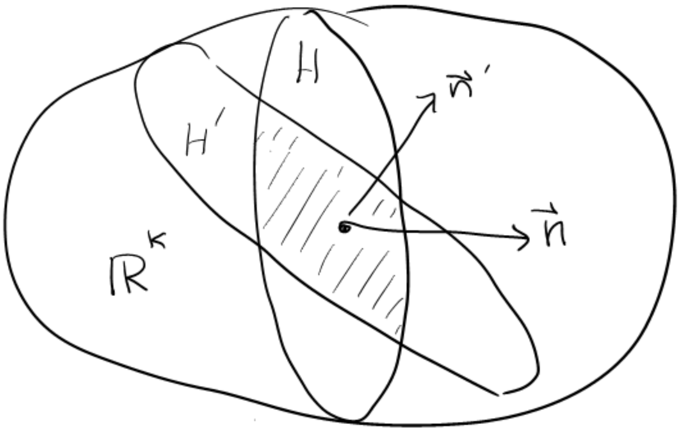
Reason

Let $\vec{a} \in H^+$ and $\vec{b} \in H^-$.
For $0 \le \lambda \le 1$, vectors $z = \lambda\vec{b} + (1-\lambda)\vec{a}$ form a line from $\vec{a}$ to $\vec{b}$.

Function $f : [0,1] \longrightarrow \mathbb{R}$
$f(\lambda) = (\lambda\vec{b} \cdot (1-\lambda)\vec{a}) \cdot \vec{n}$ is continuous, $f(0) = a \cdot \vec{n} > 0$ and $f(1) = \vec{b} \cdot n < 0$ so intermediate value theorem says $\exists \lambda_0$
$0 = f(\lambda_0) = (\lambda_0 b + (1-\lambda_0)\vec{a}) \cdot \vec{n} = 0$. Thus the line from $\vec{a}$ to $\vec{b}$ must pass through H at point $\lambda_0 b + (1-\lambda_0)\vec{a}$.

Basic Fact 2

Two hyperplanes $H, H' \subseteq \mathbb{R}^k$ intersect at a $k-2$ dimensonal subspace of $\mathbb{R}^k$.

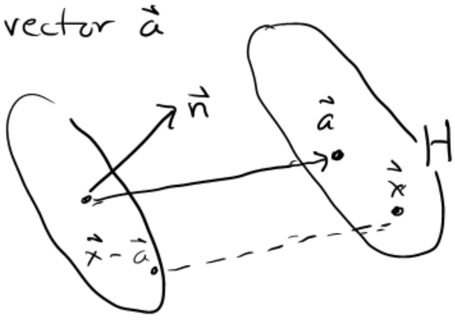


Reason $H \cap H'$ is the null space of $T : \mathbb{R}^k \to \mathbb{R}^2$
Given by $T(\vec{x}) = \begin{bmatrix} \vec{n} \cdot \vec{x} \\ \vec{n} \cdot \vec{x} \end{bmatrix}$

By rank theorem $\dim(H \cap H')$
$= \dim(\mathbb{R}^k) - \dim(\mathbb{R}^2) = n - 2$.

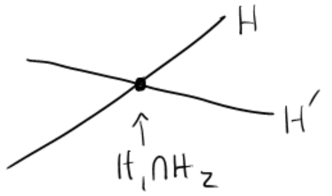
Definition In general, by hyperplane in $\mathbb{R}^k$ we mean a translate of a homogeneous hyperplane by a fixed vector $\vec{a}$

$$H = \{ \vec{x} \in \mathbb{R}^k \mid (\vec{x} - \vec{a}) \cdot \vec{n} = 0 \}$$
$$= \{ \vec{x} \in \mathbb{R} \mid \vec{x} \cdot \vec{n} = \vec{n} \cdot a \}$$

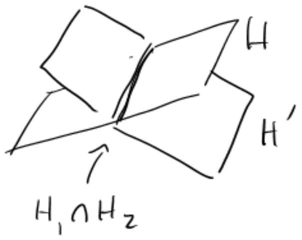


<u>Summary</u> Given hyperplanes $H, H' \subseteq \mathbb{R}^k$, either $H \cap H' = \phi$ (in which case we say they are parallel) or $H \cap H'$ is a hyperplane in each of the $k-1$ dimensional spaces H_1 & H_2.

Example $\mathbb{R}^2$

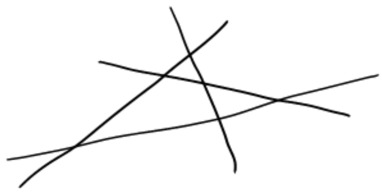


Example $\mathbb{R}^3$



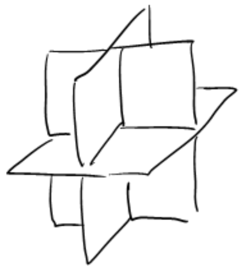
<u>Question</u> How many regions do n hyperplanes in $\mathbb{R}^k$ cut $\mathbb{R}^k$ into?

<u>Ex</u> $n=4$, $k=2$



11 regions

<u>Ex</u> $n=3$ $k=3$



8 regions

<u>Ex</u> $n=5$ $K=1$

6 regions

Note: The answer is different if some hyperplanes are parallel or more than two intersect at a point.

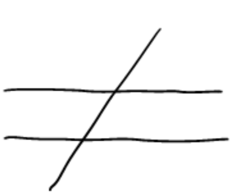
10 regions

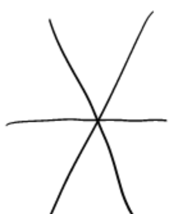
9 regions

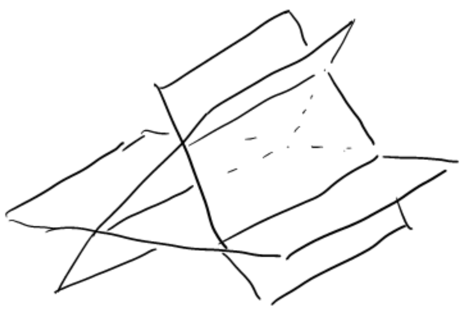
For this reason, we deal only with hyperplanes in <u>general position</u>

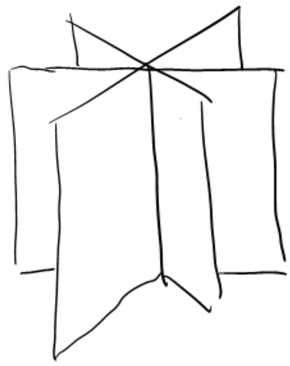
> <u>Definition</u> Hyperplanes $H_1\ H_2 \cdots H_n$ in $\mathbb{R}^k$ are in <u>general position</u> if the intersection of any l of them is a $k-l$ dimensional space for $0 \leq l \leq k$, and the intersection of more than k of them is ϕ.

<u>Examples</u>



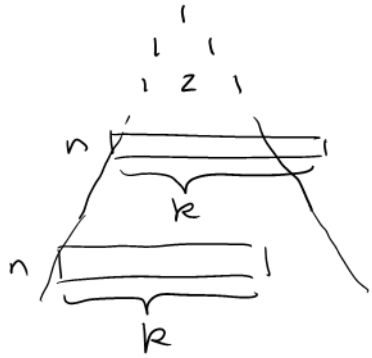






Hyperplanes <u>not</u> in general position

Now we come to our main result.

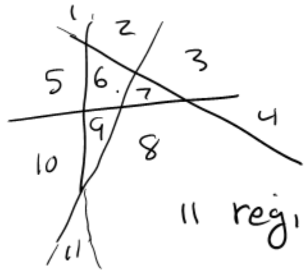


Let $h_n^{(k)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$

This is the sum of the first k terms in the nth row of Pascal's triangle. Note that k may "overshoot" the right edge of the $\triangle$. In that case the extra terms are $\binom{n}{\ell} = 0$ when $\ell > n$, and $h_n^{(k)} = 2^n$

Theorem n hyperplanes in general position in $\mathbb{R}^k$ divide $\mathbb{R}^k$ into $h_n^{(k)}$ regions

Ex $k=2$ $n=4$



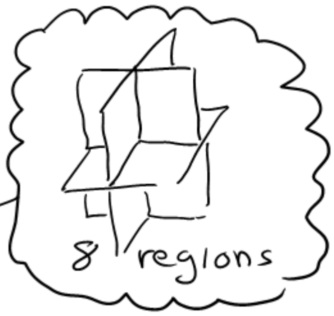
11 regions

$$h_4^{(2)} = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 1+4+6 = 11$$

$$h_1^{(4)} = \binom{1}{0} + \binom{1}{1} + \binom{1}{2} + \binom{1}{3} + \binom{1}{4} = 2$$

one hyperplane divides $\mathbb{R}^4$ into 2 regions

$$h_3^{(3)} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3 = 8$$



Proof of Theorem Fix k, so we will deal with $\mathbb{R}^k$.

Proof is induction on n

Base case:

$n=0$ 0 hyperplanes divide $\mathbb{R}^k$ into 1 region, and

$$h_0^{(k)} = \binom{0}{0} + \binom{0}{1} + \binom{0}{2} + \cdots + \binom{0}{k} = 1$$

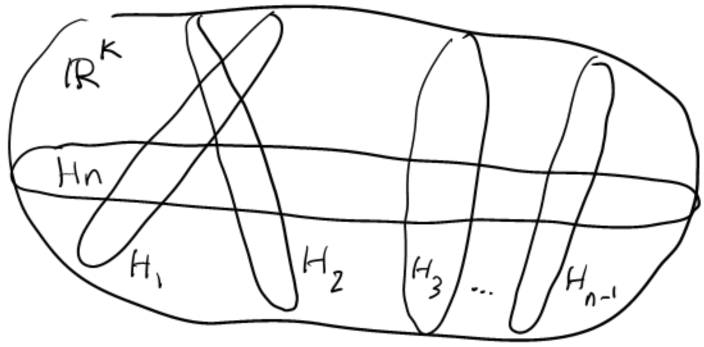
$n=1$ 1 hyperplane divides $\mathbb{R}^k$ into 2 regions and

$$h_1^{(k)} = \binom{1}{0} + \binom{1}{1} + \binom{1}{2} + \cdots + \binom{1}{k} = 2.$$

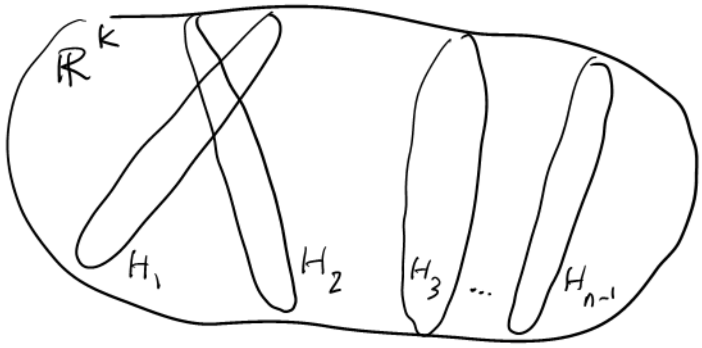
Now assume the theorem is true for n.

We will show it's also true for $n+1$.

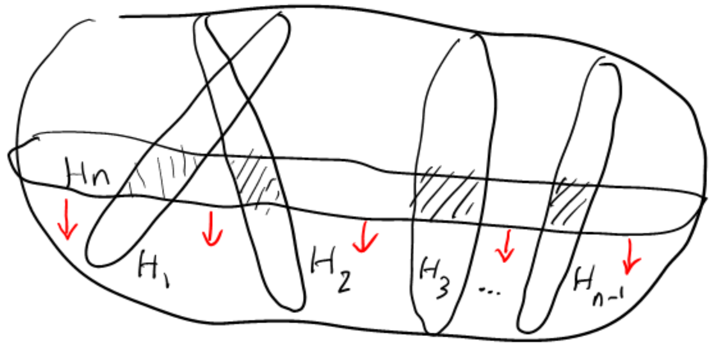
① Suppose we have n hyperplanes $H_1\ H_2\ H_3 \cdots H_n$ in general position in $\mathbb{R}^K$



② Now remove hyperplane H_n. As $n-1$ hyperplanes remain, they divide $\mathbb{R}^K$ into $h_{n-1}^{(K)}$ regions.



③ Next add H_n back. Recall $H_n \cong \mathbb{R}^{K-1}$, and each $H_i \cap H_n$ is an $(k-2)$-dimensional hyperplane in H_n. Thus hyperplanes $H_1 \cap H_n$, $H_2 \cap H_n$, $\ldots$ $H_{n-1} \cap H_n$ cut $H_n \cong \mathbb{R}^{K-1}$



into $h_{n-1}^{(K-1)}$ regions. Each one of these adds one extra region to the $h_{n-1}^{(K)}$ regions from step ②.
Thus the total number of regions in $\mathbb{R}^K$ is

$$h_{n-1}^{(K)} + h_{n-1}^{(K-1)} = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{k}$$
$$+ \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{k}$$

using $\binom{n-1}{0} = 1 = \binom{n}{0}$ and Pascal's formula $\binom{n-1}{a-1} + \binom{n-1}{a} = \binom{n}{a}$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{k}$$

$$= h_n^{(K)}$$

<u>Conclusion</u> n hyperplanes in general position cut $\mathbb{R}^K$ into $h_n^{(K)} = \sum_{i=1}^{K} \binom{n}{i}$ regions ∎