## § 8.2 Difference Sequences and Sterling Numbers

We will continue our discussion of difference sequences and use them to develop what are called Sterling numbers.

Recall that the difference operator $\Delta$ on sequences is linear: If $h_n = f_n + g_n$ for sequences $f_n$ and $g_n$, then:

$$\Delta h_n = \Delta(f_n + g_n) = \Delta f_n + \Delta g_n$$
$$\Delta^k h_n = \Delta^k(f_n + g_n) = \Delta^k f_n + \Delta^k g_n$$

$$\Delta c h_n = c\, \Delta h_n$$
$$\Delta^k c h_n = c\, \Delta^k h_n$$

This shows up in difference tables as in the following example.

$h_n = 3n^2 - n$ $\qquad\qquad$ $f_n = n^2$ $\qquad\qquad$ $g_n = n$

$$\left(\begin{array}{ccccccccc}
0 & & 2 & & 10 & & 24 & & 44 \cdots \\
& 2 & & 8 & & 14 & & 20 \cdots & \\
& & 6 & & 6 & & 6 \cdots & & \\
& & & 0 & & 0 \cdots & & & \\
& & & & 0 \cdots & & & &
\end{array}\right) = 3\left(\begin{array}{ccccccccc}
0 & & 1 & & 4 & & 9 & & 16 \\
& 1 & & 3 & & 5 & & 7 \cdots & \\
& & 2 & & 2 & & 2 \cdots & & \\
& & & 0 & & 0 \cdots & & & \\
& & & & 0 \cdots & & & &
\end{array}\right) - \left(\begin{array}{ccccccccc}
0 & & 1 & & 2 & & 3 & & 4 \cdots \\
& 1 & & 1 & & 1 & & 1 \cdots & \\
& & 0 & & 0 & & 0 \cdots & & \\
& & & 0 & & 0 \cdots & & &
\end{array}\right)$$

Recall also that the difference table for a sequence $h_n$ is entirely determined by its first row and by its zero diagonal.

$$\begin{array}{ccccccc}
3 & & 1 & & 5 & & 3 \cdots \\
\hline
& -2 & & 4 & & -2 \cdots & \\
& & 6 & & -6 \cdots & & \\
& & & -12 \cdots & & & \\
& & & \vdots & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
3 & & 1 & & 5 & & 3 \cdots \\
& -2 & & 4 & & -2 \cdots & \\
& & 6 & & -6 \cdots & & \\
& & & -12 \cdots & & & \\
& & & \ddots & & &
\end{array}$$

As we explore these ideas it will be helpful to remember that $P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$ is a degree-$k$ polynomial with variable $n$. Thus $P(n,0)\ P(n,1)\ P(n,2) \cdots$ are linearly independent functions of $n$.

Thus $\sum a_k P(n,k) = \sum b_k P(n,k) \iff a_k = b_k \ \forall k.$

As $\binom{n}{k} = \frac{1}{k!} P(n,k)$, then $\binom{n}{k}$ is a degree-$k$ polynomial in $n$ and $\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \binom{n}{3} \cdots$ are linearly independent functions of $n$.

Question: What sequence $h_n$ has zero diagonal with $p^{th}$ entry 1 and all other entries 0?

$h_n = 1 = \binom{n}{0}$

$$\begin{array}{ccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 0 & & 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 0 & & 0 & & 0 & & \\
& & & 0 & & 0 & & 0 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$h_n = n = \binom{n}{1}$

$$\begin{array}{ccccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 1 & & 1 & & 1 & & 1 & \\
& & 0 & & 0 & & 0 & & 0 & & \\
& & & 0 & & 0 & & 0 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$h_n = \binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n)$

$$\begin{array}{ccccccccccc}
0 & & 0 & & 1 & & 3 & & 6 & & 10 \\
& 0 & & 1 & & 2 & & 3 & & 4 & \\
& & 1 & & 1 & & 1 & & 1 & & \\
& & & 0 & & 0 & & 0 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$h_n = \binom{n}{3}$

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 1 & & 4 & & 10 \\
& 0 & & 0 & & 1 & & 3 & & 6 & \\
& & 0 & & 1 & & 2 & & 3 & & \\
& & & 1 & & 1 & & 1 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$h_n = \binom{n}{4}$

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 1 & & 5 \\
& 0 & & 0 & & 0 & & 1 & & 4 & \\
& & 0 & & 0 & & 1 & & 3 & & \\
& & & 0 & & 1 & & 2 & & & \\
& & & & 1 & & 1 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$\Rightarrow$

$h_n = 5\binom{n}{0} + 3\binom{n}{1} + 2\binom{n}{2} + 7\binom{n}{3} + 6\binom{n}{4}$

$$\begin{array}{ccccccccccccc}
5 & & * & & * & & * & & * & & * & & * \\
& 3 & & * & & * & & * & & * & & * & \\
& & 2 & & * & & * & & * & & * & & \\
& & & 7 & & * & & * & & * & & & \\
& & & & 6 & & * & & * & & & & \\
& & & & & 0 & & * & & & & & \\
& & & & & & 0 & & & & & &
\end{array}$$

**Theorem 8.2.2** If the zero diagonal for a sequence $h_n$ has zero diagonal $c_0\ c_1\ c_2\ c_3 \cdots c_p\ 0\ 0\ 0 \ldots$

Then $h_n = \sum_{k=0}^{p} c_k \binom{n}{k} = \sum_{k=0}^{p} \frac{c_k}{k!} P(n,k)$

Example

$$\begin{array}{ccccccccccc}
1 & & 0 & & 1 & & 4 & & 9 & & 16 \\
& -1 & & 1 & & 3 & & 5 & & 7 & \\
& & 2 & & 2 & & 2 & & 2 & & \\
& & & 0 & & 0 & & 0 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$$\begin{aligned}
h_n &= 1\binom{n}{0} - \binom{n}{1} + 2\binom{n}{2} \\
&= 1 - n + 2\frac{n(n-1)}{2} \\
&= 1 - n + n^2 - n \\
&= n^2 - 2n + 1
\end{aligned}$$

## Significant Example, leading to Sterling numbers

Write $h_n = n^p$ in the form of Theorem 8.2.2

$h_n = n^0$

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & &
\end{array}$$

$h_n = 1\,P(n,0)$

$h_n = n^1$

$$\begin{array}{ccccccccc}
0 & & 1 & & 2 & & 3 & & 4 \\
 & 1 & & 1 & & 1 & & 1 & \\
 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 0 & & & &
\end{array}$$

$h_n = 0\,P(n,0) + 1\,P(n,1)$

$h_n = n^2$

$$\begin{array}{ccccccccc}
0 & & 1 & & 4 & & 9 & & 16 \\
 & 1 & & 3 & & 5 & & 7 & \\
 & & 2 & & 2 & & 2 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 0 & & & &
\end{array}$$

$$\begin{aligned} h_n &= 0\,P(n,0) + 1\,P(n,1) + \frac{2}{2!}P(n,2) \\ &= 0\,P(n,0) + 1\,P(n,1) + 1\,P(n,2) \end{aligned}$$

$h_n = n^3$

$$\begin{array}{ccccccccc}
0 & & 1 & & 8 & & 27 & & 64 \\
 & 1 & & 7 & & 19 & & 37 & \\
 & & 6 & & 12 & & 18 & & \\
 & & & 6 & & 6 & & & \\
 & & & & 0 & & & &
\end{array}$$

$$\begin{aligned} h_n &= 0\,P(n,0) + 1\,P(n,1) + \frac{6}{2!}P(n,2) + \frac{6}{3!}P(n,3) \\ &= 0\,P(n,0) + 1\,P(n,1) + 3\,P(n,2) + 1\,P(n,3) \end{aligned}$$

$h_n = n^4$

$$\begin{array}{ccccccccc}
0 & & 1 & & 16 & & 81 & & 256 \\
 & 1 & & 15 & & 65 & & 175 & \\
 & & 14 & & 50 & & 110 & & \\
 & & & 36 & & 60 & & & \\
 & & & & 24 & & & &
\end{array}$$

$$\begin{aligned} h_n &= 0\,P(n,0) + 1\,P(n,1) + \frac{14}{2!}P(n,2) + \frac{36}{3!}P(n,3) + \frac{24}{4!}P(n,4) \\ &= 0\,P(n,0) + 1\,P(n,1) + 7\,P(n,2) + 6\,P(n,3) + 1\,P(n,4) \end{aligned}$$

**Note** $$n^p = \sum_{k=0}^{p} S(p,k)\,P(n,k) \quad \text{where} \quad S(p,k) = \frac{\left(\begin{array}{c} k^{th} \text{ entry of} \\ 0\text{-diagonal} \\ \text{for } h_n = n^p \end{array}\right)}{k!}$$

> **Definition** The numbers $S(p,k)$ are called Sterling numbers of the second kind

Example from above calculation for $h_n = n^3$,

$S(3,0) = 0 \quad S(3,1) = 1 \quad S(3,2) = 3 \quad S(3,3) = 1.$

Some Sterling numbers are easy to compute from their definition.

$$n^p = S(p,0)P(n,0) + S(p,1)P(n,1) + \cdots + S(p,p-1)P(n,p-1) + S(p,p)P(n,p)$$

$$n^p = S(p,0) + S(p,1)\,n + \cdots + \underbrace{S(p,p)\,n(n-1)(n-2)\cdots(n-p+1)}$$

If $p \neq 0$
plug in $n=0$
get $0 = 0^p = S(p,0)$

last term is the only one with an $n^p$ coefficient must be 1, so $S(p,p)=1$

Thus $\boxed{S(p,0)=0}$ and $\boxed{S(p,p)=1}$ $\Rightarrow$ $\boxed{S(0,0)=1}$

For computing other Sterling numbers we may use the following recurrence

**Theorem 8.2.4** $S(p,k) = S(p-1,k-1) + k\,S(p-1,k)$

for $1 \le k \le p-1$

[Read the proof in the text!]

From this we get a Pascal-like triangle for Sterling numbers.

| | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ |
|---|---|---|---|---|---|---|---|
| $p=0$ | 1 | | | | | | |
| $p=1$ | 0 | 1 | | | | | |
| $p=2$ | 0 | 1 | 1 | | | | |
| $p=3$ | 0 | 1 | 3 | 1 | | | |
| $p=4$ | 0 | 1 | 7 | 6 | 1 | | |
| $p=5$ | 0 | 1 | 15 | 25 | 10 | 1 | |
| $p=6$ | 0 | 1 | 31 | 90 | 65 | 15 | 1 |

**Theorem 8.2.5** $S(p,k)$ equals the number of ways to put elements of $\{1,2,3,4,\ldots p\}$ into indistinguishible boxes so that no box is empty.

## Examples

$p=0 \quad k=0: \quad S(0\,0) = 0$ [impossible to do, so 0 ways]

$p=1 \quad k=0 \quad S(1,0) = 0$

$p=1 \quad k=1 \quad S(1,1) = 1 \quad \boxed{1}$

$p=2 \quad k=0 \quad S(2,0) = 0$

$p=2 \quad k=1 \quad S(2,1) = 1 \quad \boxed{12}$

$p=2 \quad k=2 \quad S(2,2) = 1 \quad \boxed{1}\boxed{2}$

$p=3 \quad k=0 \quad S(3,0) = 0$

$p=3 \quad k=1 \quad S(3,1) = 1 \quad \boxed{123}$

$p=3 \quad k=2 \quad S(3,2) = 3 \quad \boxed{1}\boxed{23} \quad \boxed{2}\boxed{13} \quad \boxed{3}\boxed{12}$

$p=3 \quad k=3 \quad S(3,3) = 1 \quad \boxed{1}\boxed{2}\boxed{3}$

$p=4 \quad k=0 \quad S(4,0) = 6$

$p=4 \quad k=1 \quad S(4,1) = 1 \quad \boxed{1234}$

$p=4 \quad k=2 \quad S(4,2) = 7 \quad \boxed{1}\boxed{234} \quad \boxed{2}\boxed{134} \quad \boxed{3}\boxed{124} \quad \boxed{4}\boxed{123} \quad \boxed{12}\boxed{34} \quad \boxed{13}\boxed{24} \quad \boxed{14}\boxed{23}$

$p=4 \quad k=3 \quad S(4,3) = 6 \quad \boxed{1}\boxed{2}\boxed{34} \quad \boxed{1}\boxed{3}\boxed{24} \quad \boxed{1}\boxed{4}\boxed{23} \quad \boxed{2}\boxed{3}\boxed{\ } \quad \boxed{2}\boxed{4}\boxed{13} \quad \boxed{3}\boxed{4}\boxed{12}$

$p=4 \quad k=4 \quad S(4,4) = 1 \quad \boxed{1}\boxed{2}\boxed{3}\boxed{4}$

## Proof of Theorem 8.2.5

Let $S^*(p\ k)$ be the number of ways to put elements of $\{1,2,\dots p\}$ into $k$ boxes with at least one element in each box.

Then $S^*(p,0) = 0 = S(p,0)$
and $S^*(p,p) = 1 = S(p,p) \quad \boxed{1}\boxed{2}\boxed{3}\cdots\boxed{p}$

Now we just need to show $S^*(p,k)$ satisfies the same recurrence as $S(p,k)$, i.e. show $S^*(p,k) = S^*(p-1,k-1) + kS^*(p-1,k)$.

$$S^*(p,k) = S^*(p-1, k-1) + k\,S^*(p-1, k)$$

number of ways to have $p$ in a box by itself

$\boxed{p} \quad \underbrace{\boxed{\ }\ \boxed{\ }\ \cdots\ \boxed{\ }}_{k-1 \text{ boxes for } \{1,2,\dots p-1\}}$

number of ways to have $p$ in a box with at least one other number

$\underbrace{\boxed{\ }\ \boxed{\ }\ \cdots\ \boxed{\ }}_{k \text{ boxes for } \{1,2,3\dots p-1\}}$

Put $p$ in any one of the $k$ boxes.