§ 8.2 Difference Sequences and Stirling Numbers

Our main object of study is a discrete version of the derivative. Suppose we have a sequence $h_0 \; h_1 \; h_2 \; h_3 \; \ldots$ defined by a function

$$h_n = f(n)$$

We define a new sequence, the difference sequence

$$\Delta h_0 \quad \Delta h_1 \quad \Delta h_2 \quad \Delta h_3 \quad \ldots$$

defined as

$$\Delta h_n = h_{n+1} - h_n$$

$$\Delta f(n) = f(n+1) - f(n) = \frac{f(n+1) - f(n)}{1} = \lim_{h \to 0} \frac{f(n+h) - f(n)}{h}$$

h is discrete, so $h \to 0$ means $h = 1$

Example

	h_0		h_1		h_2		h_3		h_4		h_5		h_6		h_7	
	3		5		10		12		13		16		20		30	$\ldots$
		2		5		2		1		3		4		10		$\ldots$
		Δh_0		Δh_1		Δh_2		Δh_3		Δh_4		Δh_5		Δh_6		

Definitions

2^{nd} order difference sequence $\Delta^2 h_n = \Delta(\Delta h_n)$

3^{rd} order difference sequence $\Delta^3 h_n = \Delta(\Delta^2 h_n)$

$\vdots$

n^{th} order difference sequence $\Delta^n h_n = \Delta(\Delta^{n-1} h_n)$

Example $h_n = 2^n$

h_n	1	2	4	8	16	32	64	$\ldots$
Δh_n	1	2	4	8	16	32	$\ldots$	
$\Delta(\Delta h_n) = \Delta^2 h_n$	1	2	4	8	16	$\ldots$		
$\Delta(\Delta^2 h_n) = \Delta^3 h_n$	1	2	4	8	$\ldots$			

"difference table" for $h_n = 2^n$

Taking difference sequence of geometric sequence $h_n = 2^n$ does not change it. (Like $D_x[e^x] = e^x$)

Ex $h_n = 3^n$

1	3	9	27	81	243	...
	2	6	18	54	162	...
		4	12	36	108	...
			8	24	72	...

"difference table" for $h_n = 3^n$

$$\Delta h_n = 3^{n+1} - 3^n = 3^n(3-1) = 2\cdot 3^n = 2h_n$$

$$\Delta^2 h_n = 2\cdot 3^{n+1} - 2\cdot 3^n = 3^n(2\cdot 3 - 2) = 4\cdot 3^n = 4h_n$$

$$\Delta 3^n = 2\cdot 3^n$$

Ex $h_n = a^n$

$$\Delta h_n = a^{n+1} - a^n = a^n(a-1)$$

$$\Delta a^n = a^n(a-1)$$

like $D_x[a^x] = a^x \ln(a)$ so $a-1$ is a "discrete" version of $\ln(a)$

Ex $h_n = n^3 + n^2 + n + 1$

h_n	1	4	15	40	85	156	259 ...	← row 0
Δh_n	3	11	25	45	71	103 ...		← row 1
$\Delta^2 h_n$	8	14	20	26	32 ...			← row 2
$\Delta^3 h_n$	6	6	6	6 ...				← row 3
$\Delta^4 h_n$	0	0	0 ...					← row 4
$\Delta^5 h_n$	0	0 ...						← row 5
								⋮

"zero diagonal"

Theorem 8.2.1

If $h_n = f(n) = a_p n^p + a_{p-1} n^{p-1} + \dots a_1 n + a_0$, then $\Delta^{p+1} h_n = 0$, and $\Delta^k h_n = 0$ for all $k \geq p+1$.

[Read proof in text]

The analogy with derivatives continues with analogs of

$$D_x[f(x) \pm g(x)] = D_x[f(x)] \pm D_x[g(x)]$$

and

$$D_x[c f(x)] = c D_x[f(x)]$$

<u>Proposition</u> If $h_n = f_n + g_n$ then $\Delta h_n = \Delta f_n + \Delta g_n$

<u>Proof</u>

$$\begin{aligned} \Delta h_n &= h_{n+1} - h_n \\ &= (f_{n+1} + g_{n+1}) - (f_n + g_n) \\ &= (f_{n+1} - f_n) + (g_{n+1} - g_n) = \Delta f_n + \Delta g_n \end{aligned}$$

<u>Proposition</u> $\Delta c h_n = c \Delta h_n$

<u>Proof</u>

$$\begin{aligned} \Delta c h_n &= c h_{n+1} - c h_n \\ &= c(h_{n+1} - h_n) = c \Delta h_n. \end{aligned}$$

Note that a difference table is determined entirely by its 0^{th} row, <u>but also</u> by its zero diagonal.

a_0 $\square \leftarrow a_0 + a_1$

a_1 $\square \leftarrow a_1 + a_2$

a_2 $\square \leftarrow a_2 + a_3$

a_3 $\square \leftarrow a_3 + a_2$

$\ddots$

<u>Example</u>

```
3   3   4   11   36   108
  0   1   7   25   72
    1   6   18   47
      5   12   29
        7   17
          10   ⋱
            ⋱
```