Chapter 8 Special Counting Sequences

§8.1 Catalan Numbers

Recall the following formula from §7.6:

$$\begin{pmatrix}\text{Number of ways to divide an} \\ (n+1)\text{-gon into } n-1 \text{ triangles by} \\ \text{joining } n-2 \text{ diagonals}\end{pmatrix} = \frac{1}{n}\binom{2n-2}{n-1}$$

Definion The <u>Catalan numbers</u> are the members of the sequence $C_0\ C_1\ C_2\ C_3 \ldots$ defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Therefore C_n is the number of ways to divide an $(n+2)$-gon into Δ's by connecting diagonals.

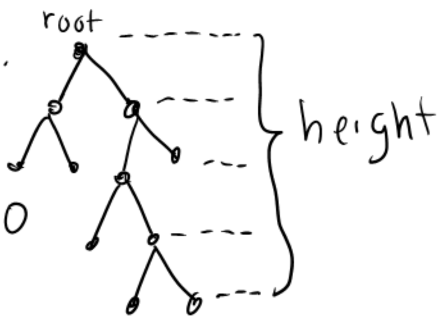
$C_0 = 1$	$C_3 = 5$	$C_6 = 132$
$C_1 = 1$	$C_4 = 14$	$C_7 = 429$
$C_2 = 2$	$C_5 = 42$	$C_8 = 1430$

Text develops recurrence relation (with $C_0 = 1$):

$$C_n = \frac{4n-2}{n+1} C_{n-1}$$

Today we will look at some seemingly unrelated combinatorial objects that Catalan numbers count.

One class of such objects are <u>binary trees</u>.



A <u>binary tree</u> is a tree where one vertex (called the root) has degree 2 or 0 and all others have degree 3 or 1.
A vertex of degree less than 2 is called a <u>leaf</u>.

Proposition A binary tree with n leaves has $2n-2$ edges.

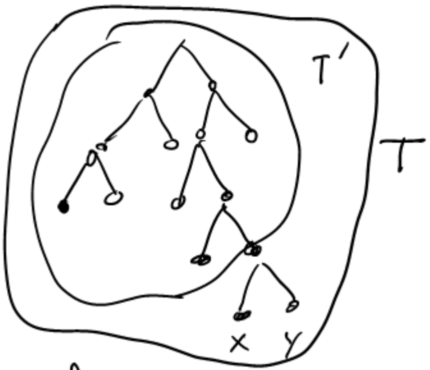
Proof (Induction on n)

True if $n=1$: • 1 leaf and $2\cdot1-2=0$ edges.

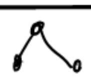
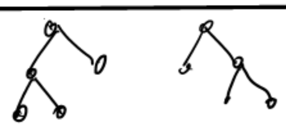
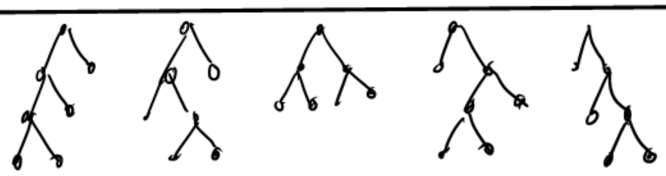
True if $n=2$:  2 leaves and $2\cdot2-2=2$ edges.

Assume true for all trees with fewer than $n>2$ leaves.

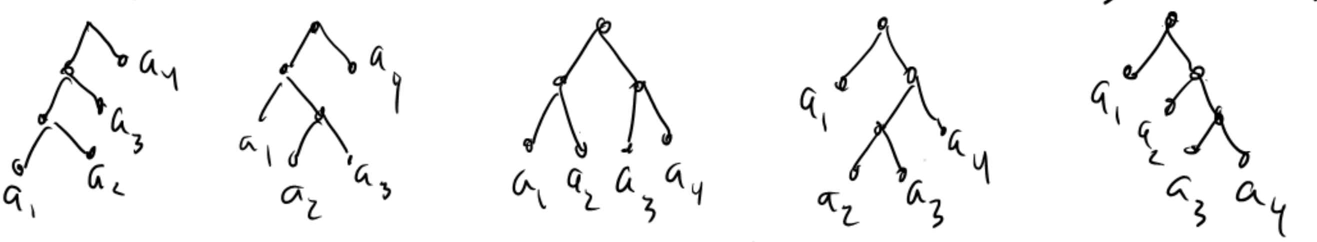
Take a binary tree T with n leaves. Necessarily its height is greater than 1. Let x, y be two leaves at maximum height. Let $T' = T - \{x,y\}$, so T' has $n-1$ leaves. By inductive hypothesis it has $2(n-1)-2 = 2n-4$ edges, so T has $(2n-4)+2 = 2n-2$ edges.



Let b_n = # of binary trees with $n+1$ leaves

n	Binary Trees	b_n
0		$b_0 = 1$
1		$b_1 = 1$
2		$b_2 = 2$
3		$b_3 = 5$

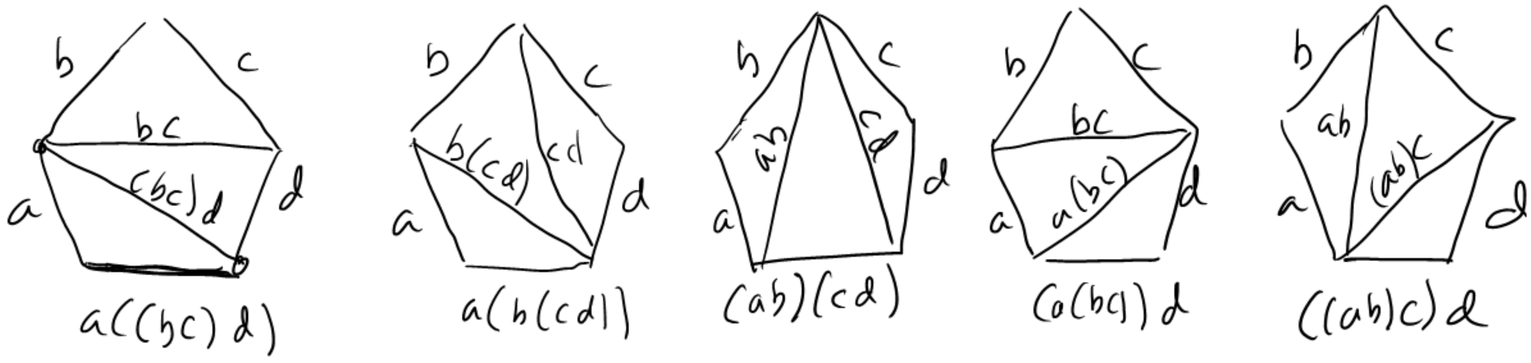
We will see shortly that $b_n = C_n$. But first note that any binary tree on $n+1$ leaves is associated with a <u>multiplication scheme</u> for numbers $a_1\, a_2\, a_3 \cdots a_{n+1}$



$(((a_1a_2)a_3)a_4) \quad ((a_1(a_2a_3))a_4 \quad (a_1a_2)(a_3a_4) \quad a_1((a_2a_3)a_4) \quad a_1(a_2(a_3a_4))$

Now we claim that the number m_n of multiplication schemes on $n+1$ numbers $a_1\, a_2 \dots a_{n+1}$ equals the number of divisions of an $(n+2)$-gon into triangles by diagonals.

Ex $\{a_1, a_2, a_3, a_4\} = \{a, b, c, d\}$



<u>Conclusion</u>

$$\begin{pmatrix}\text{binary}\\ \text{trees with}\\ n+1 \text{ leaves}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{multiplication}\\ \text{schemes on}\\ n+1 \text{ numbers}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{subdivisions}\\ \text{of } (n+2)\text{-gon}\\ \text{into } \triangle\text{'s}\end{pmatrix}$$

$$b_n \quad = \quad m_n \quad = \quad C_n$$

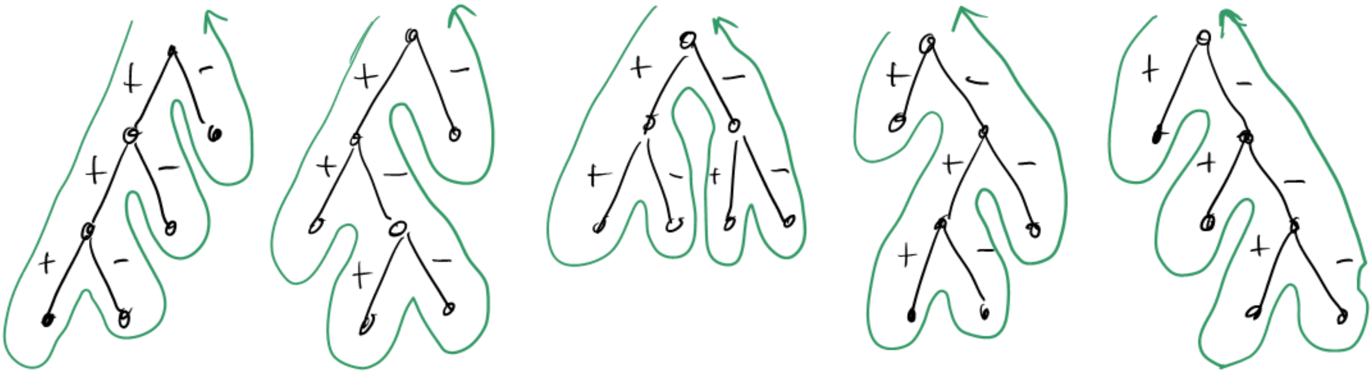
Theorem 8.1.1 The number of sequences $a_1\, a_2\, a_3 \dots a_{2n}$ with $a_i = \pm 1$ and $\sum_{i=1}^{k} a_i \geq 0$ for $1 \leq k \leq 2n$, and $\sum_{i=1}^{2n} a_i = 0$. is C_n

$n=1$	1		$\longleftarrow C_1 = 1$ of these
$n=2$	$1\ {-1}\ 1\ {-1}$	$1\ 1\ {-1}\ {-1}$	$\longleftarrow C_2 = 2$ of these
$n=3$	$1\ 1\ 1\ {-1}\ {-1}\ {-1}$	$1\ {-1}\ 1\ 1\ {-1}\ {-1}$	
	$1\ 1\ {-1}\ {-1}\ 1\ {-1}$	$1\ {-1}\ 1\ {-1}\ 1\ {-1}$	$\longleftarrow C_3 = 5$ of These.
	$1\ 1\ {-1}\ 1\ {-1}\ {-1}$		

<u>Proof</u> We will argue (informally) that the number of such sequences $a_1\, a_2 \dots a_{2n}$ equals the number b_n of binary trees with $n+1$ leaves.

Such a tree has $2(n+1)-2 = 2n$ edges, i.e. an edge for each term in $a_1\ a_2\ a_3 \cdots a_{2n}$.

Each such tree corresponds to a sequence $a_1\ a_2\ a_3 \cdots a_{2n}$ of n 1's and n -1's whose partial sums are not negative. To get the sequence, traverse the tree counterclockwise from the root. As you encounter an edge for the first time it corresponds to a $+1$ if it is and a -1 if it is

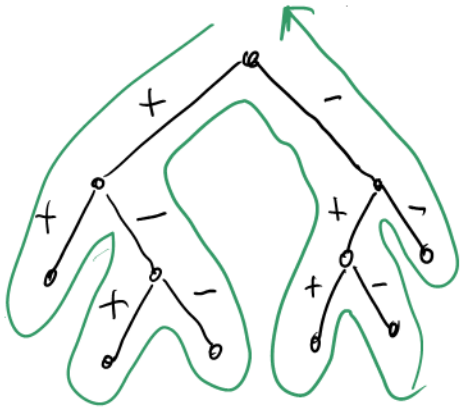


$1+1+1-1-1-1$ $\quad$ $1+1-1+1-1-1$ $\quad$ $1+1-1-1+1-1$ $\quad$ $1-1+1+1-1-1$ $\quad$ $1-1+1-1+1-1$

Conversely, any sequence $a_1\ a_2\ a_3 \cdots a_{2n}$ of the stated form corresponds to a unique binary tree with $n+1$ leaves and $2n$ edges.

Example:

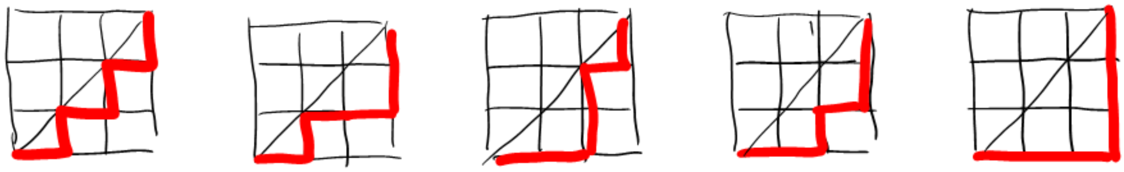
$1\ \ 1\ \ -1\ \ 1\ \ -1\ \ -1\ \ 1\ \ 1\ \ -1\ \ -1$



Conclusion # of $a_1\ a_2 \cdots a_{2n}$ with $a_i = \pm 1$, $\sum_{i=1}^{2n} a_i = 0$ and $\sum_{i=1}^{k} a_i \geq 0$ is $b_n = C_n$.

Here is another Catalan connection. Consider a $n \times n$ grid

A route is acceptable if it goes from A to B moving only right or up in each step.

n	routes on $n \times n$ grid, below diagonal	# of routes
1		$1 = C_1$
2		$2 = C_2$
3		$5 = C_3$

Number of acceptable routes on $n \times n$ grid is C_n.

<u>Reason:</u> Traverse route from A to B, each $\rightarrow$ is $+1$ and each $\uparrow$ is -1.

The text defines the pseudo-Catalan numbers as $c_0^*\ c_1^*\ c_2^* \ldots$ where $c_n^* = n!\,C_{n-1}$

# of multiplication schemes on n numbers

Thus C_n^* gives the number of multiplication schemes of $a_1\ a_2 \ldots a_n$, where we are allowed to arrange the numbers in any order.

<u>Ex</u> $C_3 = 3!\,C_2 = 6 \cdot 2 = 12$

$a(bc)$	$a(cb)$	$b(ac)$	$b(ca)$	$c(ab)$	$c(ba)$
$(ab)c$	$(ac)b$	$(ba)c$	$(bc)a$	$(cb)a$	$(cb)a$