§7.4 Generating Functions

The key idea of generating functions is that answers to many combinatorial questions are found among the coefficients of polynomials.

Ex $(1+x)(1+x)(1+x+x^2) = \underbrace{1x^0+3x^1+4x^2+3x^3+1x^4+0x^5+0x^6+\cdots}$

$$\underset{\underset{x^a}{\uparrow}}{\begin{Bmatrix}x^0\\x^1\end{Bmatrix}}\ \underset{\underset{x^b}{\uparrow}}{\begin{Bmatrix}x^0\\x^1\end{Bmatrix}}\ \underset{\underset{x^c}{\uparrow}}{\begin{Bmatrix}x^0\\x^1\\x^2\end{Bmatrix}}$$

coefficient of x^n is # of products $x^ax^bx^c$ with $a+b+c=n$, $0\le a,b\le 1$ and $0\le c\le 2$ = # of n-combos of $\{\underset{\infty}{a}, \underset{\infty}{b}, \underset{\infty}{c}\}$ with 0 or 1 a's and 0 or 1 b's and 0, 1, or 2 c's

For instance if $n=2$ they are $\{a,b\}$, $\{a,c\}$, $\{b,c\}$ $\{c,c\}$

Ex $(\underbrace{1+x+x^2+x^3+\cdots}_{x^a})(\underbrace{1+x+x^2+x^3+\cdots}_{x^b})(\underbrace{1+x+x^2+x^3+\cdots}_{x^c})$

$$= 1x^0+3x^1+6x^2+10x^3+15x^4+\cdots$$

coefficient of x^n is # of products $x^ax^bx^c = x^n$ = # of non-neg. integer solutions of $a+b+c=n$

Ex $(x^0+x^2+x^4+x^6+\cdots)(x^2+x^4) = \underbrace{0x^0+0x^1+1x^2+0x^3+2x^4+0x^5+\cdots}$

Coefficient of x^n is number of ways to get $x^n = x^ax^b$ with a even, $b=2,4$ = # of ways to put n balls in 2 boxes where 1st box has even # of balls and 2nd has 2 or 4

Ex $(1+x)^k = \binom{k}{0}x^0+\binom{k}{1}x^1+\binom{k}{2}x^2+\cdots+\binom{k}{k}x^k+0x^{k+1}+0x^{k+2}\cdots$

coefficient of x^n equals # of n-element subsets of k-element set

We now develop this idea.

<u>Definition</u> Given an infinite sequence $h_0 \ h_1 \ h_2 \ h_3 \cdots$, its <u>generating</u> <u>function</u> is the infinite series

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + \cdots$$

Here are examples of some of the more common generating functions. Look at these as building blocks for others.

<u>Ex</u> $1, 1, 1, 1, 1, \cdots, \underset{\uparrow h_n}{1}, 0, 0, 0, 0, 0, 0, \cdots$

$$g(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \longleftarrow$$

formula for sum of 1st n terms of geometric series with ratio x

<u>Ex</u> $1, 1, 1, 1, 1, \cdots, 1, \cdots$

$$g(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1 - x} \longleftarrow$$

converges provided $|x| < 1$, but we don't bother with issues of convergence

<u>Ex</u> $\binom{5}{0} \ \binom{5}{1} \ \binom{5}{2} \ \binom{5}{3} \ \binom{5}{4} \ \binom{5}{5} \ \binom{5}{6} \ \binom{5}{7} \cdots$

$$g(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 + 0x^6 + 0x^7 + \cdots = (1+x)^5$$

<u>Ex</u> $1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \cdots$

$$g(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots = e^x$$

<u>Key points</u>

- Generating functions are infinite power series.
- Sometimes there is a tidy formula for the infinite sum.
- We never plug anything into the x, so we don't have to worry about convergence. The x is just a "placeholder"
- $h_0 h_1 h_2 h_3 \cdots$ are the coefficients of Taylor expansion of the generating function

Basic formulas neeeded for working with generating functions

① Given: $\begin{cases} f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots \end{cases}$

Then $f(x)g(x) =$

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_1)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots$$

and n^{th} term is $(a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0)x^n$

② $$\frac{1+x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n$$

③ $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

④ $$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$$

⑤ $(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots$ ← Newtons binomial formula

⑥ $$(1+x^m)^\alpha = 1 + \binom{\alpha}{1}x^m + \binom{\alpha}{2}x^{2m} + \binom{\alpha}{3}x^{3m} + \binom{\alpha}{4}x^{4m} + \cdots$$

⑦ $$(1+x)^{-k} = 1 + \binom{-k}{1}x + \binom{-k}{2}x^2 + \binom{-k}{3}x^3 + \binom{-k}{4}x^4 + \cdots + \binom{-k}{n}x^n + \cdots$$

$$\frac{1}{(1+x)^k} = 1 - \binom{k}{1}x^1 + \binom{k+1}{2}x^2 - \binom{k+2}{3}x^3 + \binom{k+3}{4}x^4 - \cdots \pm \binom{n+k-1}{n}x^n + \cdots$$

$$\binom{-k}{n} = \frac{(-k)(-k-1)(-k-2)(-k-n+1)}{n!} = (-1)^k\binom{k+n-1}{n}$$

⑧ $$\frac{1}{(1-x)^k} = 1 + \binom{k}{1}x + \binom{k+1}{2}x^2 + \binom{k+2}{3}x^3 + \binom{k+3}{4}x^4 + \cdots + \binom{n+k-1}{n}x^n + \cdots$$

Ex Find generating function for The number of integer solutions of $a+b+c+d=n$.

$$(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)$$

↑ in expansion x^n term is sum of all terms $x^a x^b x^c x^d$ with $a+b+c+d=n$. Thus coefficient of x^n equals number of integer solutions of $a+b+c+d=n$.

Ans $$g(x) = (1+x+x^2+\dots)^4 = \left(\frac{1}{1-x}\right)^4 = \frac{1}{(1-x)^4} = \sum_{n=0}^{\infty} \binom{n+3}{n} x^n$$

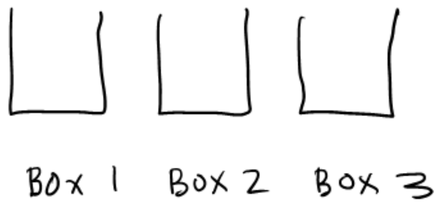
Ex Find generating function for number of solutions of $a+b+c+d=n$ where a is even, b is a multiple of 7 $0 \le c \le 5$.

Ans.

$$g(x) = (1+x^2+x^4+x^6+\dots)(1+x^7+x^{14}+x^{21}+\dots)(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+\dots)$$

$$= \frac{1}{1-x^2}\,\frac{1}{1-x^7}\,\frac{1-x^6}{1-x}\,\frac{1}{1-x} = \frac{1-x^6}{(1-x^2)(1-x^7)(1-x)^2}$$

Ex You have an an unlimited supply of identical marbles Find The generating function for the number of ways to put n marbles into 3 boxes, so that Box 1 has at least 4 marbles, Box 2 has and odd # of marbles and Box 3 has no more Than 4 marbles.



$$g(x) = (x^4+x^5+x^6+\dots)(x+x^3+x^5+x^7+\dots)(1+x+x^2+x^3+x^4)$$

$$= x^4(1+x+x^2+\dots)\,x(1+x^2+x^4+x^6+\dots)\,\frac{1-x^5}{1-x}$$

$$= x^5\,\frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1-x^5}{1-x} = \frac{x^5(1-x^5)}{(1-x^2)(1-x)^2}$$

Ex Find a generating function for the number of n-combinations of $S = \{\infty \cdot A, \infty \cdot B, \infty \cdot C\}$ where # of A's is multiple of 6, # of B's is between 0 and 5 and there are 0 or 1 C's.

$$g(x) = (1 + x^6 + x^{12} + x^{18} + \cdots)(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x)$$

$$= \frac{1}{1 - x^6} \cdot \frac{1 - x^{5+1}}{1 - x}(1 + x) = \frac{1}{1 - x}(1 + x)$$

$$= (1 + x + x^2 + x^3 + \cdots)(1 + x)$$

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots$$

Thus, for any $n > 0$, there are just 2 such n-combos.

$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
B	BB	BBB	BBBB	BBBBBB	AAAAAA	AAAAAAB
C	BC	BBC	BBBC	BBBBC	BBBBBC	AAAAAAC