§7.3 Non-Homogeneous Recurrence Relations.

Homogeneous linear recurrence relation: $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \dots + a_k h_{n-k}$

Non-homogeneous linear recurrence relation: $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \dots + a_k h_{n-k} + f(n)$

Non-zero term may be constant or a function of n

Ex $f(n) = 5n+7$ $\quad f(n) = 1$

$f(n) = 3^n$

Non-homogeneous recurrence relations can be difficult to solve. We will look at some of the simpler types.

They also arize naturally in combinatorial problems.

Example Tower of Hanoi Puzzle.

There are n circular disks of n sizes, large to small. Initially all disks are stacked on one pole, in order of size

Objective: Move all disks to another pole, never putting a larger disk atop a smaller.

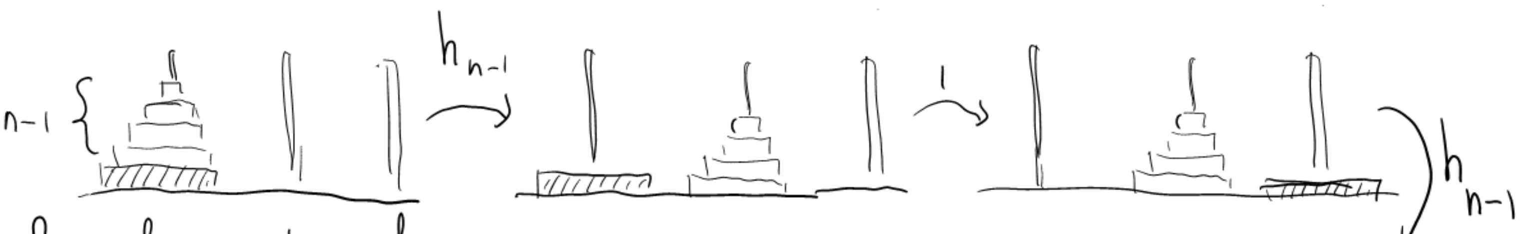
If there are n disks, how many moves does it take to solve?

Let h_n = (# of moves to solve n-disk problem)

$h_0 = 0$

$h_1 = 1$

$h_2 = 3$



$h_n = h_{n-1} + 1 + h_{n-1}$

$$\boxed{h_n = 2h_{n-1} + 1}$$

Non homogeneous recurrence relation

How would we solve such a relation?
Here is an idea that will work.

The <u>homogeneous part</u> of $h_n = 2h_{n-1} + 1$ is $h_n = 2h_{n-1}$

general solution $h_n = c2^n$

A <u>particular solution</u> of $h_n = 2h_{n-1} + 1$ is $h_n = -1$

particular solution

Let's try adding these two solutions: $h_n = c2^n - 1$

Then
$$h_n = c2^n - 1 = 2(c2^{n-1} - 1) + 1$$
$$= 2h_{n-1} + 1$$

Thus $h_n = c2^n - 1$ satisfies recurrence relation $h_n = 2h_{n-1} + 1$

For Tower of Hanoi Problem initial value is
$(n=0)$ $h_0 = c2^0 - 1 = 0 \Rightarrow c = 1$

Thus solution of Tower of Hanoi problem is $\boxed{h_n = 2^n - 1}$

This approach may seem nutty, but today we will see that it works.

But to keep things simple we will concentrate on those of form

$$h_n = ah_{n-1} + f(n) \qquad *$$

The relation $h_n = ah_{n-1}$ is called the <u>homogeneous part</u> of $*$. Its general solution is $h_n = ca^n$.

Key Point

Consider $h_n = a h_{n-1} + f(n)$

Take <u>any</u> particular solution $\boxed{h_n = \varphi(n)}$ $*$

$$h_n = a h_{n-1} + f(n)$$

$$\varphi(n) = a\,\varphi(n-1) + f(n)$$

<u>Homogeneous part</u> $g_n = a g_{n-1}$ has solution $\boxed{g_n = c a^n}$ $**$

Now let $h_n = c a^n + \varphi(n)$ be sum of homogen. & particular sol'ns.

$$\begin{aligned} h_n &= c a^n + \varphi(n) \\ &= c a^n + a\varphi(n-1) + f(n) \\ &= a\left(c a^{n-1} + \varphi(n-1)\right) + f(n) \\ &= a h_{n-1} + f(n) \end{aligned}$$

That is, the sum of $**$ and $*$ satisfies given recurr. rel.

<u>Strategy</u>: To solve a non-homogeneous recurrence relation, find <u>one</u> particular solution and add it to the solution of the homogeneous part.

Success in applying this strategy hinges on being able to find the particular solution $\varphi(n)$. The following chart may be helpful

$f(n)$	$\varphi(n)$
r	d
$rn+s$	$dn+e$
rn^2+sn+t	fn^2+dn+e
d^n	pd^n

Ex Solve $h_n = 3h_{n-1} + 2n$

with initial values 1 5 19 ...

Homogeneous part has solution $h_n = c\,3^n$

For particular solution try $\varphi(n) = dn + e$

$$\varphi(n) = 3\varphi(n-1) + 2n$$

$$dn + e = 3\big(d(n-1) + e\big) + 2n$$

$$dn + e = 3dn - 3d + 3e + 2n$$

$$0 = (2d+2)n + \underbrace{2e - 3d}$$

$$2d + 2 = 0 \qquad\qquad 2e - 3d = 0$$

$$d = -1 \qquad\qquad 2e + 3 = 0$$

$$e = -\tfrac{3}{2}$$

Particular Solution $\varphi(n) = -n - \frac{3}{2}$

Check:

$$\varphi(n) \stackrel{?}{=} 3\varphi(n-1) + 2n$$

$$-n - \tfrac{3}{2} \stackrel{?}{=} 3\left(-(n-1) - \tfrac{3}{2}\right) + 2n$$

$$-n - \tfrac{3}{2} \stackrel{?}{=} -3n + 3 - \tfrac{9}{2} + 2n$$

$$-n - \tfrac{3}{2} \stackrel{?}{=} -n - 3/2$$

General Solution: $h_n = c\,3^n - n - \frac{3}{2}$

$(n=0)$ $h_0 = c\,3^0 - 0 - \frac{3}{2} = 1$

$$c = 1 + 3/2 = \tfrac{5}{2}$$

$$h_n = \tfrac{5}{2}3^n - n - \tfrac{3}{2}$$

$$h_0 = \tfrac{5}{2}3^0 - 0 - \tfrac{3}{2} = \tfrac{2}{2} = 1$$

$$h_1 = \tfrac{5}{2}3^1 - 1 - \tfrac{3}{2} = \tfrac{15}{2} - \tfrac{3}{2} - 1 = 5$$

$$h_2 = \tfrac{5}{2}3^2 - 2 - \tfrac{3}{2} = \tfrac{45}{2} - \tfrac{3}{2} - 2 = 21 - 2 = 19$$

Ex Solve $h_n = 2h_{n-1} + 3^n$ with initial value $h_0 = 4$

Homogeneous solution: $h_n = c\,2^n$

Particular solution $\varphi(n) = p\,3^n$

$$\varphi(n) = 2\varphi(n-1) + 3^n$$
$$p\,3^n = 2\,p\,3^{n-1} + 3^n$$
$$3p = 2p + 3$$
$$p = 3$$

Thus $\varphi(n) = 3 \cdot 3^n = 3^{n+1}$

Check that this satisfies recurrence relation:

$$\varphi(n) \stackrel{?}{=} 2\,\varphi(n-1) + 3^n$$
$$3^{n+1} \stackrel{?}{=} 2\,3^n + 3^n$$
$$3^{n+1} = 3\,3^n \quad \checkmark$$

General solution:

$$h_n = c\,2^n + 3^{n+1}$$

$(n=0)$ $h_0 = c\,2^0 + 3^{0+1} = 4$

$$c + 3 = 4$$
$$c = 1$$

Answer $\boxed{h_n = 2^n + 3^{n+1}}$