Section 7.2 Linear Homogeneous Recurrence Relations

Definition A sequence $h_0\ h_1\ h_2\ h_3 \ldots$ satisfies a linear recurrence relation of order k if there are quantities $a_1\ a_2 \ldots a_k$ and b_0 for which

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + a_3 h_{n-3} + \cdots + a_k h_{n-k} + b_0$$

whenever $n > k$. Values a_i and b_0 may depend on n (i.e. they may be functions of n) or constant, and $a_k \neq 0$. If $b_0 = 0$ the recurrence relation is said to be homogeneous, otherwise it is non-homogeneous.

Ex Fibonacci Sequence .... $h_n = h_{n-1} + h_{n-2}$

0 1 1 2 3 5 8 13 ...

$$h_n = \underset{\substack{\uparrow\\ a_1}}{1} h_{n-1} + \underset{\substack{\uparrow\\ a_2 = a_k}}{1} \cdot h_{n-2} + \underset{\substack{\uparrow\\ b_0}}{0}$$

Homogeneous, linear recurrence relation of order 2

Ex Geometric Sequence

1 2 4 8 16 32 ... . $h_n = 2 \cdot h_{n-1}$

$$= \underset{\substack{\uparrow\\ a_1}}{2} h_{n-1} + \underset{\substack{\uparrow\\ b_0}}{0}$$

Homogeneous linear recurrence relation of order 1

Ex 1, 1, 1, 10, 161 .... $h_n = \underset{\substack{\uparrow\\ a_1}}{n^2} h_{n-1} + \underset{\substack{\uparrow\\ a_2}}{1} \cdot h_{n-3}$

Homogeneous, linear of order 3

Ex 2 2 2 11 162 .... $h_n = n^2 h_{n-1} + h_{n-3} + 1$

Non-homogeneous, linear, order 3

There are non-linear recurrence relations, but we will not consider them.

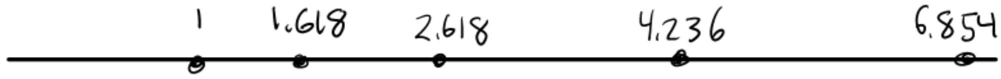
Ex 1, 2, 4, 32 ... $\quad h_n = h_{n-1}^2 \cdot h_{n-2}$

Non-linear, order 2.

A <u>solution</u> of a recurrence relation is a function $h_n = F(n)$ for which the sequence $F(0), F(1), F(2), F(3) \cdots$ satisfies the recurrence relation

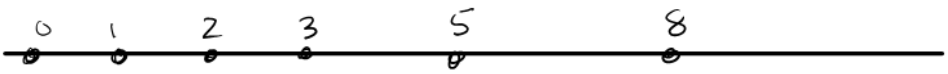
<u>Example</u> Fibonacci recurrence relation $h_n = h_{n-1} + h_{n-2}$ has solutions

$$h_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$



$$h_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$h_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$



$$h = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

<u>Note</u> Each Solution above is a power series or a linear combination of power series. This is typical.

<u>Key Observation</u>

How to find a solution of a linear homogeneous recurrence rel.

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + a_3 h_{n-3} + \cdots + a_k h_k$$

① Write it as $h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0$

② Make polynomial $x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_k x^{n-k} = 0$

③ Let q be a non-zero root of this polynomial

Then $h_n = q^n$ is a solution of the recurrence relation.

Reason: Just plug $x = q$ into ① & compare with ②

- Different roots give different solutions.
- Roots could be integers, rationals, irrationals or complex

Next lets look at a definition and theorem that use this idea.

Definition Given recurrence relation $h_n - a_1 h_{n-1} - a_2 h_{n-2} - \dots - a_k h_{n-k} = 0$, form

$$x^n - a_1 x^{n-1} - a_2 x^{n-2} - \dots - a_k x^{n-k} = 0$$

$$x^{n-k}(x^k - a_1 x^{k-1} - a_2 x^{k-2} - \dots - a_k) = 0$$

$$\boxed{x^k - a_1 x^{k-1} - a_2 x^{k-2} - \dots - a_k = 0}$$

This is called the characteristic equation of the recurrence relation. Any root q of it gives a solution $h_n = q^n$.

Theorem 7.2.1

Consider a homogeneous linear recurrence relation

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \dots - a_k h_{n-k} = 0.$$

Suppose the characteristic equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \dots - a_k = 0$$

has distinct roots $q_1\ q_2\ q_3 \dots q_k$. Then

$$h_n = c_1 q_1^n + c_2 q_2^n + \dots + c_k q_k^n \qquad *$$

is a solution of the recurrence relation for any constants c_i. Moreover, given any initial values $h_0\ h_1\ h_2 \dots h_k$ values of c_i can be found so so that $*$ produces these initial values.

Example Consider sequence $1, 1, 5, 13, 41, 121, \dots$ given by $h_n = 2h_{n-1} + 3h_{n-2}$.

Find a formula for h_n (i.e. solve the recurrence relation subject to initial values $h_0 = 1$ $h_1 = 1$

Solution

$$h_n = 2h_{n-1} + 3h_{n-2}$$

Looking for non-zero root so this division is OK

$$h_n - 2h_{n-1} - 3h_{n-2} = 0$$

$$x^n - 2x^{n-1} - 3x^{n-2} = 0$$

$$\frac{1}{x^{n-2}}\left(x^n - 2x^{n-1} - 3x^{n-2}\right) = 0\,\frac{1}{x^{n-2}}$$

$$x^2 - 2x - 3 = 0$$

characteristic equation

$$(x-3)(x+1) = 0$$

$$x = 3 \qquad\qquad x = -1$$

By Theorem 7.2.1 we get the following general solution.

$$h_n = c_1(3)^n + c_2(-1)^n$$

Now we just need to find c_1 & c_2 that give the right initial conditions. This requires:

$$(n=0)\quad c_1\cdot 3^0 + c_2(-1)^0 = 1 \qquad c_1 + c_2 = 1$$

$$(n=1)\quad c_1\cdot 3^1 + c_2(-1)^1 = 1 \qquad 3c_1 - c_2 = 1$$

Solving by row reduction $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -4 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1/2 \end{array}\right]$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right]$$

Therefore $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$ so

$$\boxed{h_n = \frac{1}{2}3^n + \frac{1}{2}(-1)^n} \leftarrow \text{ANSWER}$$

Check

$$h_0 = \frac{1}{2}3^0 + \frac{1}{2}(-1)^0 = \frac{1}{2} + \frac{1}{2} = 1$$

$$h_1 = \frac{1}{2}3 + \frac{1}{2}(-1) = \frac{2}{2} = 1$$

$$h_2 = \frac{1}{2}3^2 + \frac{1}{2}(1)^2 = \frac{9}{2} + \frac{1}{2} = 5$$

$$h_3 = \frac{1}{2}3^3 + \frac{1}{2}(-1)^3 = \frac{27}{2} - \frac{1}{2} = 13$$

$$h_4 = \frac{1}{2}3^4 + \frac{1}{2}(-1)^4 = \frac{81}{2} + \frac{1}{2} = 41$$