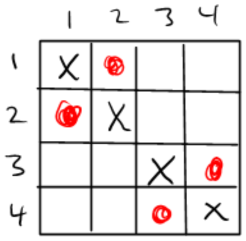
Section 6.3 Derangements.

Question How many ways can 4 non-attacking rooks be placed on a 4x4 chessboard so that no rook occupies a diagonal?



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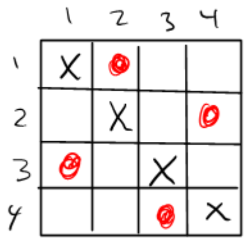

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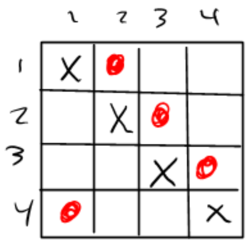

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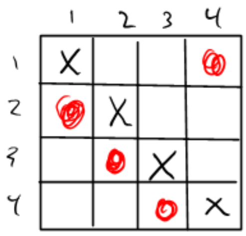

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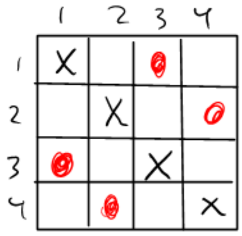

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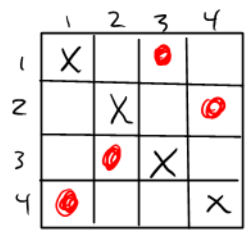

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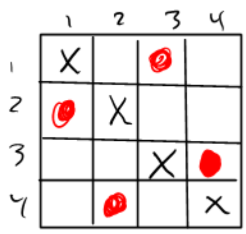

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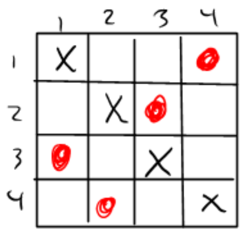

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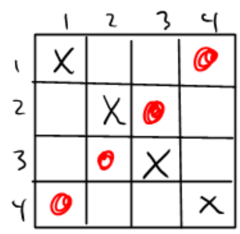


4321

There are 9 such arrangements. They correspond to the permutations of 1234 in which no number appears in its natural position (i.e. i^{th} entry of permutation does not equal i) as shown above.

Note a rook arrangement corresponds to the permutation $i_1 i_2 i_3 i_4$ where the rook in column 1 is in row i_1, the rook in column 2 is in row i_2, etc.

Permutations in wich no entry appears in its natural position are called *derangements*

A <u>derangement</u> of a set S is a permutation of S in which none of its elements are in their original position.
<u>Notation</u> D_n = # of derangements of an n-element set.

<u>Examples</u>

S	permutations of S	derangements of S	D_n
$\{1\}$	1	none	$D_1 = 0$
$\{1,2\}$	12 21	21	$D_2 = 1$
$\{1,2,3\}$	123 132 213 231 312 321	231 312	$D_3 = 2$
$\{1,2,3,4\}$	1234 1243 ... 1324	2143 3142 4123 2341 3412 4312 2413 3421 4321	$D_4 = 9$

Inclusion - Exclusion gives an easy formula for D_n

Let $S = \{1,2,3,\dots n\}$ and let $\mathcal{U}$ be set of all permutations of S, so $|\mathcal{U}| = n!$

Also $A_1 \subseteq S$ is permutations where 1 is in 1^{st} position
$A_2 \subseteq S$ " " " " 2 " " 2^{nd} "
$\vdots$
$A_n \subseteq S$ " " " " n ... n^{th} "

Then $|A_i| = (n-1)!$ for each i.
$|A_i \cap A_j| = (n-2)!$
$|A_i \cap A_j \cap A_k| = (n-3)!$ etc.

Then set of all derangements of S is $\overline{A}_1 \cap \overline{A}_2 \cap \dots \cap \overline{A}_n$

so $D_n = |\overline{A}_1 \cap \overline{A}_2 \cap \dots \cap \overline{A}_n|$

$= |\mathcal{U}| - \sum|A_i| + \sum|A_i \cap A_j| - \sum|A_i \cap A_j \cap A_k| + \cdots$

$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots \pm \binom{n}{n}(n-n)!$

$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots \pm \frac{n!}{n!} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

$$\boxed{\textbf{Theorem} \quad D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right)}$$

$$D_1 = 1!\left(1 - \frac{1}{1!}\right) = 0$$
$$D_2 = 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) = 1$$
$$D_3 = 3!\left(1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6}\right) = 2$$
$$D_4 = 4!\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 9$$
$$D_5 = 5!\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44$$
$$\vdots$$

Example

$$\frac{5!}{D_5} = \frac{120}{44} = 2.\overline{72}$$
$$\approx 2.71828\ldots = e$$

Recall: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Thus: $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

Therefore $D_n \approx n!\,e^{-1}$ so $n! \approx e D_n$ and moreover $\lim_{n\to\infty} \frac{n!}{D_n} = e$

i.e. between $\frac{1}{2}$ and $\frac{1}{3}$ of permutations of an n-element set are derangements.

Notice: $D_n \approx n!\,e^{-1} = n\left[(n-1)!\,e^{-1}\right] \approx n D_{n-1}$

$$\boxed{\text{Exact Formula: } D_n = n D_{n-1} + (-1)^n}$$

Proof:

$$\begin{aligned} n D_{n-1} + (-1)^n &= n\,(n-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right) + (-1)^n \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right) + (-1)^n \frac{n!}{n!} \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!} + (-1)^n \frac{1}{n!}\right) \\ &= D_n \end{aligned}$$

We also have the following

$$\boxed{\text{Formula: } D_n = (n-1)\left(D_{n-2} + D_{n-1}\right)}$$

(Verify it yourself or read proof in text.)

Section 6.4 Permutations with Forbidden Positions

Motivational Example

$S = \{1,2,3,4,5\}$ ← We denote a permutation of S as $i_1\ i_2\ i_3\ i_4\ i_5$

$X_1 = \{1,3,4,5\}$

$X_2 = \{1,2,3\}$

$X_3 = \{5\}$

$X_4 = \phi$

$X_5 = \{1,2,3\}$

$$P(X_1 X_2 X_3) = \{ i_1 i_2 i_3 i_4 i_5 \mid i_1 \notin X_1 \quad i_2 \notin X_2 \quad i_3 \notin X_3 \quad i_4 \notin X_4 \quad i_5 \notin X_5 \}$$

= set of permutations of S where i_k is forbidden to be in X_k

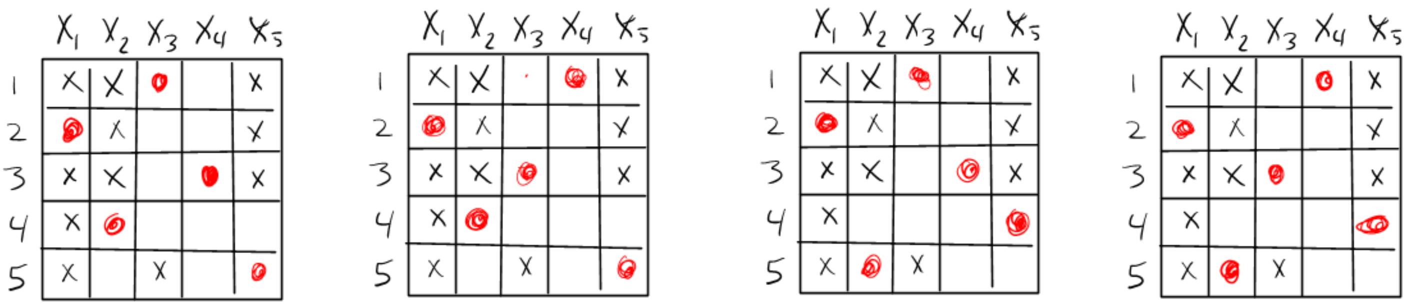
$= \{24135,\ 24315,\ 25134,\ 25314\}$

We can visualize these permutations by drawing a 5x5 grid and in each column i crossing out the squares in X_i.

	X_1	X_2	X_3	X_4	X_5
1	x	x			x
2		x			x
3	x	x			x
4	x				
5	x		x		

Note: We use the transpose of Brualdi's digrams

The x's represent "forbidden squares". The 4 permutations in $P(X_1 \cdots X_5)$ encode the 4 ways we can place non-attacking rooks on the board, avoiding forbidden squares.

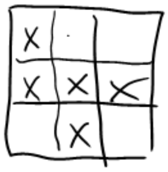


24135   24315   25134   25314

Summary $P(X_1 X_2 X_3 X_4 X_5) = \{31425,\ 31452,\ 41325,\ 41352\}$

Notation $\rho(X_1 X_2 X_3 X_4 X_5) = |P(X_1 X_2 X_3 X_4 X_5)| = 4$

Example $S = \{1,2,3\}$ $X_1 = \{1,2\}$ $X_2 = \{2,3\}$ $X_3 = \{2\}$



Then $P(X_1 X_2 \cdots X_n) = \phi$ and $p(X_1 X_2 X_3) = 0$

Example $S = \{1,2,3,\dots n\}$ and $X_i = \phi$ for $1 \le i \le n$

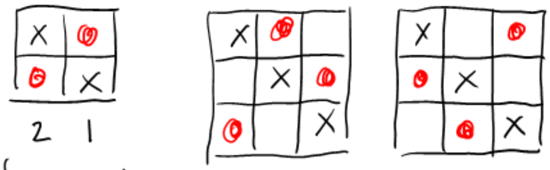
Then $P(X_1 X_2 \cdots X_n)$ is set of all permutations of S,

so $p(X_1 X_2 X_3 \dots X_n) = n!$

Example $S = \{1, 2, 3, \dots, n\}$ $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $\dots$ $X_n = \{n\}$

Then $P(X_1 X_2 \cdots X_n)$ is set of derangements of S, and $p(X_1 X_2 \cdots X_n) = D_n$

In this sense, derangements correspond to the ways to place non-attacking rooks on a board, avoiding the diagonal



$\underbrace{2 \quad 1}_{\uparrow}$ $\qquad \underbrace{3\ 1\ 2 \qquad 2\ 3\ 1}$

$D_2 = 1$ $\qquad D_3 = 2$

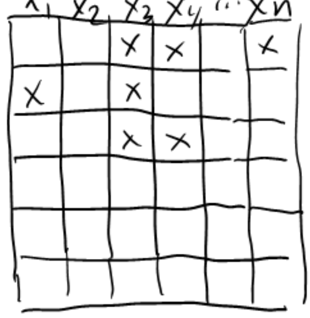
General Problem

Given $S = \{1, 2, 3, \dots, n\}$ and $X_1\ X_2\ X_3 \cdots X_n \subseteq S$, let

$P(X_1 X_2 \cdots X_n)$ be permutations of S with i^{th} entry not in X_i, $1 \le i \le n$

Find $p(X_1\ X_2 \cdots X_n)$

Solution Let U be set of all permutations of $\{1,2,3,\dots n\}$, so $|U| = n!$



A_1 = Permutations with rook in forbidden position in col. 1

A_2 = Permutations with rook in forbidden position in col. 2

$\vdots$

A_n = Permutations with rook in forbidden position in col. n

Then by inclusion-exclusion

$$p(X_1 \cdots X_n) = |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|$$

$$= |U| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots \pm |A_1 \cap \cdots \cap A_n|$$

To work this out you need to calculate the individual su

Note $|A_1| = |X_1|(n-1)!$

and $|A_i| = |X_i|(n-1)!$

By multiplication principle, put a r in any of the $|X_1|$ forbidden positio in row 1. Then there are $(n-1)!$ positions for the remaining $n-1$ rooks in the remaining rows

Thus $\sum |A_i| = \sum |X_i|(n-1)!$

$= \left(\sum |X_i|\right)(n-1)!$

$= r_1 (n-1)!$ where $r_1 = \sum |X_i| = \left(\begin{array}{c}\text{total number}\\ \text{of forbidden}\\ \text{positions}\end{array}\right) = \left(\begin{array}{c}\text{number of ways to}\\ \text{place 1 rook on}\\ \text{a forbidden position}\end{array}\right)$

Similarly $\sum |A_i \cap A_j| = r_2 (n-2)!$ where $r_2 = \left(\begin{array}{c}\text{number of ways to}\\ \text{place 2 rooks on}\\ \text{forbidden positions}\end{array}\right)$

$\sum |A_i \cap A_j \cap A_k| = r_3 (n-3)!$ where $r_3 = \left(\begin{array}{c}\text{number of ways to}\\ \text{place 3 rooks on}\\ \text{forbidden positions}\end{array}\right)$

$\vdots$

etc.

$$\boxed{\text{Thus } p(X_1 \dots X_n) = n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + \cdots + r_n}$$

Formula only useful if it's easy to compute the r_i's !!

<u>Example</u> $S = \{1,2,3,4,5\}$

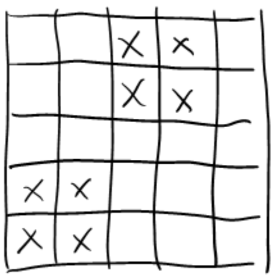
$X_1 = \{4,5\}$

$X_2 = \{4,5\}$

$X_3 = \{1,2\}$

$X_4 = \{1,2\}$

$X_5 = \emptyset$



$r_1 = 8$

$r_2 = 4\cdot 4 + 2 + 2 = 20$

$r_3 = 2\cdot 4 + 4\cdot 2 = 16$

$r_4 = 4$

$r_5 = 0$

$p(X_1 \dots X_5) = 5! - 8\cdot 4! + 20\cdot 3! - 16\cdot 2! + 4\cdot 1! + 0\cdot 0!$