Section 6.2 Combinations with Repetition.

Inclusion - Exclusion can be used to resolve some issues that were previously problematic for us. For example recall that we have the following formula involving r-combinations of multisets for which each element has rep. # ∞

> **Theorem 3.5.1** Suppose $S = \{ \underset{\infty}{a_1}, \underset{\infty}{a_2}, \underset{\infty}{a_3}, \dots, \underset{\infty}{a_k} \}$
> Then the number of r-combinations of S is $\binom{r+k-1}{r}$

$= \binom{r+k-1}{k-1}$

Question

How many r-combinations are there of $T = \{ \underset{n_1}{a_1} \ \underset{n_2}{a_2} \cdots \underset{n_k}{a_k} \}$?

The answer is slightly more involved than when the repetition numbers of each element is ∞, but can be answered with inclusion - exclusion

Solution Let $\mathcal{U}$ be set of r-combinations of $S = \{ \underset{\infty}{a_1}, \underset{\infty}{a_2}, \underset{\infty}{a_3}, \dots, \underset{\infty}{a_k} \}$

$A_1 \subseteq \mathcal{U}$ is set of r-combinations with more than n_1 a_1's

$A_2 \subseteq \mathcal{U}$ is set of r-combinations with more than n_2 a_2's

$\vdots$

$A_k \subseteq \mathcal{U}$ is set of r-combinations with more than n_k a_k's

$$|\mathcal{U}| = \binom{r+k-1}{k-1}$$

$$|A_1| = \binom{r-(n_1+1)+k-1}{k-1}$$

To make A_i, first grab n_i+1 a_i's. Next select $r-(n_i+1) = r-n_i-1$ elements from S

$$\vdots$$

$$|A_k| = \binom{r-(n_k+1)+k-1}{k-1}$$

$$|A_i \cap A_j| = \binom{r-(n_i+1)-(n_j+1)+k-1}{k-1}$$

$$|A_i \cap A_j \cap A_\ell| = \binom{r-(n_i+1)-(n_j+1)-(n_\ell+1)+k-1}{k-1} \quad \text{etc.}$$

Answer to Question

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k| = |\mathcal{U}| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_\ell| + \cdots$$

Example How many non-negative integer solutions are there to $u+v+w+x+y+z = 20$?

We already know how to solve this. Each solution is a 20-combination of the multiset $S = \{\infty \cdot u, \infty \cdot v, \infty \cdot w, \infty \cdot x, \infty \cdot y, \infty \cdot z\}$ so each solution is encoded as $* \cdots * \mid * \cdots * \mid * \cdots * \mid * \cdots * \mid * \cdots * \mid * \cdots *$

Answer: $\binom{20+6-1}{20} = \binom{25}{20} = \dfrac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{5!} = \boxed{53{,}130}$

Example How many non-negative integer solutions are there to $u+v+w+x+y+z = 20$ where each variable is between 0 & 8?

Solution Let $\mathcal{U}$ be the set of non-negative integer solutions to the equation with no restriction on the variables. Then $|\mathcal{U}| = \binom{25}{20}$ by the previous example.

Let A_u be solutions for which $u > 8$

A_v " " " " " " $v > 8$

A_w " " " " " " $w > 8$

$\vdots$

A_z " " " " " $z > 8$

Then $|A_u| = |A_v| = |A_w| = \cdots = |A_z| = \binom{20-9+6-1}{20-9} = \binom{16}{11}$

Also $|A_u \cap A_v| = \binom{20-9-9+6-1}{20-9-9} = \binom{7}{2} = |A_u \cap A_x| = |A_w \cap A_z|$, etc

And $A_u \cap A_v \cap A_w = \phi$ and similarly for the intersections of any 3 or more of the sets A_*

The answer to our question is thus

$|\mathcal{U}| - \sum |A_*| + \sum |A_* \cap A_{**}| \pm$ (all other terms 0)

$= \binom{25}{20} - \binom{6}{1}\binom{16}{11} + \binom{6}{2}\binom{7}{2} = 53130 - 6 \cdot \dfrac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{120} + 15 \cdot 21$

$= 53130 - 26208 + 315 = \boxed{27237}$