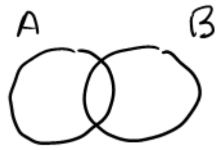
Chapter 6 Inclusion - Exclusion

§6.1 The Inclusion - Exclusion Principle

The Inclusion - Exclusion principle is a generalization of the Addition Principle.

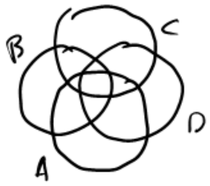
<u>Addition Principle</u> Given sets $A_1, A_2, \ldots, A_m$ with $A_i \cap A_j = \phi$ for all indices i, j,
$|A_1 \cup A_2 \cup \ldots A_m| = |A_1| + |A_2| + \ldots |A_m|$

The Inclusion - Exclusion Principle adapts this to the situation in which the sets are not necessarily disjoint.



$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



$$\begin{aligned} |A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D| \end{aligned}$$

<u>Inclusion - Exclusion Principle</u> Given sets $A_1\ A_2\ A_3 \ldots A_m$,

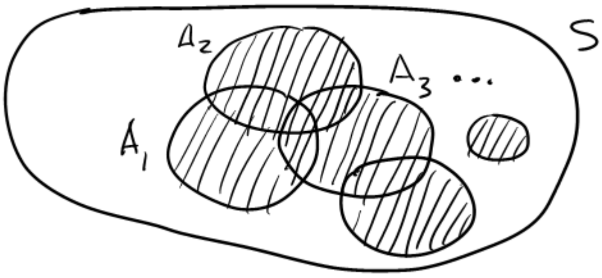
$$|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \ldots \pm |A_1 \cap A_2 \cap \ldots \cap A_m|$$

<u>Special Case</u> Suppose $A_1\ A_2 \ldots A_m$ are sets for which

$$|A_i| = \alpha_1 \quad \forall i$$
$$|A_i \cap A_j| = \alpha_2 \quad \forall i, j.$$
$$|A_i \cap A_j \cap A_k| = \alpha_3 \quad \forall i j k$$
$$\vdots$$
$$|A_1 \cap A_2 \cap \ldots \cap A_m| = \alpha_m$$

Then $|A_1 \cup A_2 \cup \ldots \cup A_m| = \binom{m}{1}\alpha_1 - \binom{m}{2}\alpha_2 + \binom{m}{3}\alpha_3 - \binom{m}{4}\alpha_4 + \ldots \pm \binom{m}{m}\alpha_m$

Typical Situation



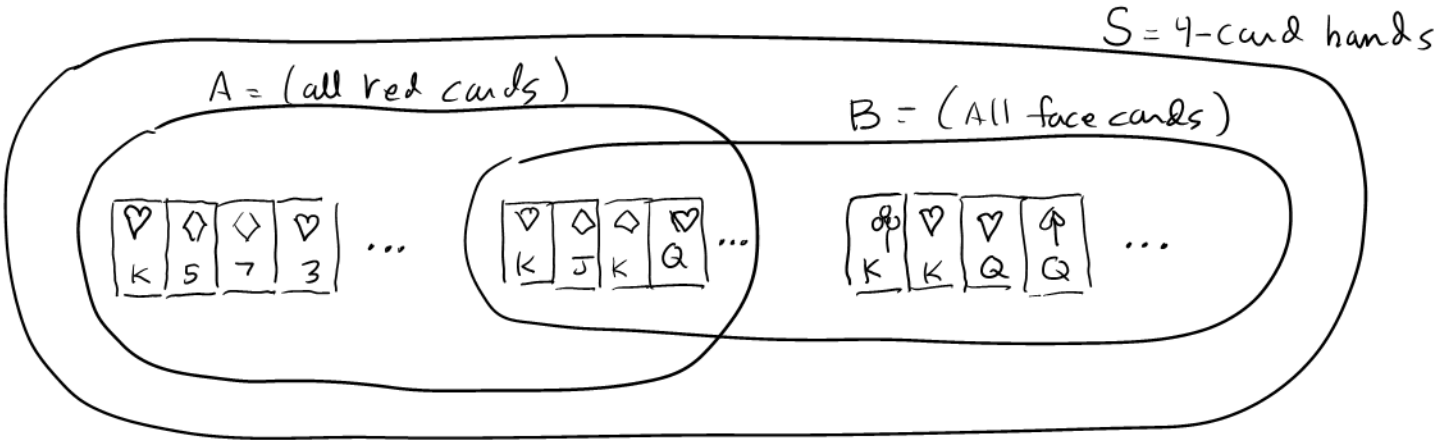
S = A set of things

$P_1 \; P_2 \; \cdots \; P_m$ are properties that things in S might possess.

$A_i = \{ x \in S \mid x \text{ has property } P_i \}$

Inclusion-Exclusion formula gives $|A_1 \cup A_2 \cup \cdots \cup A_m|$, i.e the number of things in S that have at least one of the properties P_i.

Example A 4-card hand is dealt from a 52-card deck. How many such hands are there for which all cards are red or all cards are face-cards?



$$|A| = \binom{26}{4} = \frac{26 \cdot 25 \cdot 24 \cdot 23}{4 \cdot 3 \cdot 2} = 26 \cdot 25 \cdot 23 = 14950$$

$$|B| = \binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2} = 11 \cdot 5 \cdot 9 = 495$$

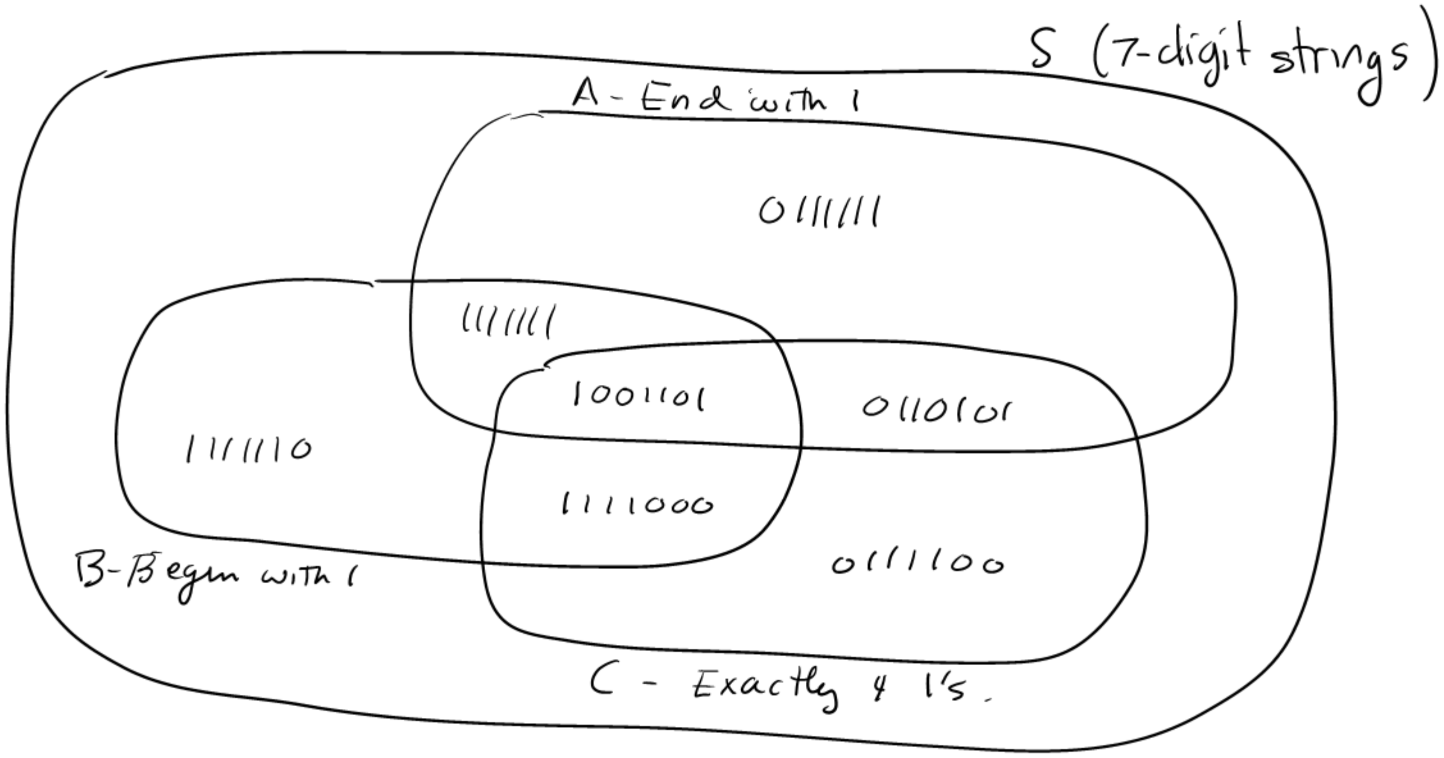
$$|A \cap B| = \binom{6}{4} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2} = 15$$

Answer:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 14950 + 495 - 15 = \boxed{15{,}430}$$

Example: How many 7-digit binary strings begin with 1, end with 1, or have exactly four 1's?



Answer:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 2^6 + 2^6 + \binom{7}{4} - 2^5 - \binom{6}{3} - \binom{6}{3} + \binom{5}{2}$$

$$= 64 + 64 + \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2} - 32 - \frac{6\cdot 5\cdot 4}{3\cdot 2} - \frac{6\cdot 5\cdot 4}{3\cdot 2} + \frac{5\cdot 4}{2}$$

$$= 128 + 35 - 32 - 20 - 20 + 10$$

$$= 101$$

Next we're going to look at a slightly different formulation of Inclusion-Exclusion. But first recall

DeMorgan's Laws: When $A, B, C \ldots$ are subsets of a universal set S,

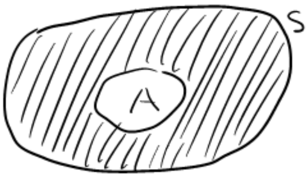
$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C} \qquad \overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

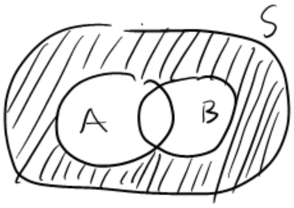
$$\vdots \qquad \vdots$$

The flip side of Inclusion - Exclusion

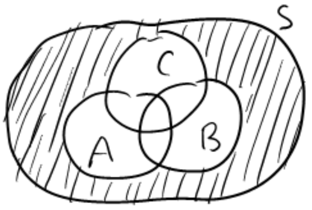
It is often useful to count things not in the union of the A_i



$$|\bar{A}| = |S| - |A|$$



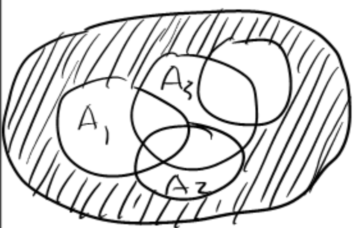
$$|\overline{A \cup B}| = |S| - |A \cup B| = |S| - |A| - |B| + |A \cap B|$$
$$\parallel$$
$$|\bar{A} \cap \bar{B}|$$



$$|\overline{A \cup B \cup C}| = |S| - |A \cup B \cup C| = |S| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$
$$\parallel$$
$$|\bar{A} \cap \bar{B} \cap \bar{C}|$$

Inclusion - Exclusion Principle

Suppose $A_1, A_2, \ldots, A_m \subseteq S$, where each A_i consists of the elements of S that have some property P_i. Then $\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m \subseteq S$ is set of things in S having none of the properties P_i. Moreover



$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| = |\overline{A_1 \cup A_2 \cup \cdots \cup A_m}|$$
$$= |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots \pm |A_1 \cap A_2 \cap \cdots \cap A_m|$$

Further, if it happens that

$$\begin{aligned} |A_i| &= \alpha_1 && \forall i \\ |A_i \cap A_j| &= \alpha_2 && \forall i,j \\ |A_i \cap A_j \cap A_k| &= \alpha_3 && \forall i,j,k \\ &\vdots \\ |A_1 \cap A_2 \cap \cdots \cap A_m| &= \alpha_m \end{aligned}$$

then
$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| = |S| - \alpha_1 \binom{n}{1} + \alpha_2 \binom{n}{2} - \alpha_3 \binom{n}{3} + \cdots \pm \alpha_n \binom{n}{n}$$

Example

How many 6-digit numbers do not have 3 consecutive digits?

[In this problem, lets include such strings as 004321 as 6-digit numbers, i.e. insert leading zeros when needed]

Example We count numbers such as 004321, 225511
but not 000321, 321000, 222223, etc.

Solution Let S be the set of all six-digit numbers, so $|S| = 10^6$.

Let $A_0 \in S$ be numbers with 3 consecutive 0's
$A_1 \subseteq S$ " " " " " 1's
$A_2 \subseteq S$ " " " " " 2's
$\vdots$
$A_9 \subseteq S$ " " " " " 9's

Looking at A_0 it contains the numbers

$000\boxed{9}\boxed{10}\boxed{10}$, $\boxed{9}000\boxed{9}\boxed{10}$, $\boxed{10}\boxed{9}000\boxed{9}$ $\boxed{10}\boxed{10}\boxed{9}000$

not 0 not 0 not 0 not 0 not 0 not 0

$0000\boxed{9}\boxed{10}$ $\boxed{9}0000\boxed{9}$

not 0 not 0 not 0

$00000\boxed{9}$ $\boxed{9}00000$

not 0 not 0

000000

Thus $|A_0| =$
$900 + 810 + 810 + 900$
$+ 90 + 90$
$+ 9 + 9$
$+ 1 = 3619$

Similarly $|A_0| = |A_1| = \cdots = |A_9| = 3619$

Note $A_0 \cap A_1 = \{000111, 111000\}$, etc so $A_i \cap A_j = 2$

Also $A_i \cap A_j \cap A_k = \phi$

Answer $|S| - \binom{10}{1}3619 + \binom{10}{2}\cdot 2 - \binom{10}{3}0 + \cdots$

$= 1000000 - 36190 + 10\cdot 9 = \boxed{963900 \text{ numbers}}$