Section 5.5 The Multinomial Theorem

Recall that the Binomial Theorem gives the coefficients of

$$(x+y)^n$$

The theorem says this equals $\binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}x^0 y^n$

Today's Goal: Multinomial Theorem that gives coefficients of:

$$(x+y+z)^n$$

$$(w+x+y+z)^n$$

$$(x_1 + x_2 + \dots + x_k)^n \qquad \text{etc.}$$

For example, direct computation shows that

$$\begin{aligned}(x+y+z)^3 = x^3 + y^3 + z^3 &+ 3x^2y + 3x^2z \\ &+ 3xy^2 + 3y^2z \\ &+ 3xz^2 + 3yz^2 \quad + 6xyz\end{aligned}$$

Each term has form $\mu x^a y^b z^c$ with $a+b+c=3$ where the coefficient μ is some constant. So we are asking: How can we find the coefficients μ in general? The Multinomial Theorem will answer this question.

They key to answering this question is Theorem 4.3.2

> **Theorem 4.3.2** Suppose S is the multiset $S = \{ \underset{n_1}{a_1}\ \underset{n_2}{a_2} \dots \underset{n_k}{a_k} \}$ so $|S| = n_1 + n_2 + \dots + n_k$
>
> Then the number of permutations of S is $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$

Example $S = \{x,x,x,y,y,y,y,z\}$ has $\dfrac{8!}{3!\,4!\,1!} = 280$ permutations

> Notation If $n = n_1 + n_2 + \dots n_k$ then $\begin{pmatrix} & & n & & \\ n_1 & n_2 & n_3 & \dots & n_k \end{pmatrix} = \dfrac{n!}{n_1!\, n_2! \dots n_k!}$

Example $\begin{pmatrix} & 8 & \\ 3 & 4 & 1 \end{pmatrix} = \dfrac{8!}{3!\,4!\,1!} = 280$

Example

$\{a+b=n, \text{ so } n-b=a\}$

Coefficient of $x^a y^b$ in $(x+y)^n$ is $\binom{n}{b} = \frac{n!}{b!(n-b)!} = \frac{n!}{b!\,a!} = \binom{n}{a\ \ b}$

In general, we will see that multinomial coefficients have form $\binom{n}{n_1\ n_2 \cdots n_k}$

A few examples will guide our way

Ex $(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy = x^2 + 2xy + y^3$

permutations of $\{x,x\}$: $\binom{2}{2\ 0} = 1$ of these

Permutations of $\{x,y\}$: $\binom{2}{1\ 1} = 2$ of these

permutations of $\{y,y\}$: $\binom{2}{0\ 2} = 1$ of these

Ex $(x+y+z)^3 = (x+y+z)(x+y+z)(x+y+z) =$

$= xxx$	$+\ yyy$	$+\ zzz$	$+\ xxy$	$+\ xxz$	$+\ yyz$	$+\ xyy$	$+\ xzz$	$+\ yzz$	$+\ xyz$
			$+\ xyx$	$+\ xzx$	$+\ yzy$	$+\ yxy$	$+\ zxz$	$+\ zyz$	$+\ xzy$
			$+\ yxx$	$+\ zxx$	$+\ zyy$	$+\ yyx$	$+\ zzx$	$+\ zzy$	$+\ yxz$
									$+\ yzx$
									$+\ zxy$
									$+\ zyx$
$\binom{3}{3\ 0\ 0}$	$\binom{3}{0\ 3\ 0}$	$\binom{3}{0\ 0\ 3}$	$\binom{3}{2\ 1\ 0}$	$\binom{3}{2\ 0\ 1}$	$\binom{3}{0\ 2\ 1}$	$\binom{3}{1\ 2\ 0}$	$\binom{3}{1\ 0\ 2}$	$\binom{3}{0\ 1\ 2}$	$\binom{3}{1\ 1\ 1}$
$=1$	$=1$	$=1$	$=3$	$=3$	$=3$	$=3$	$=3$	$=3$	$=6$

Therefore $(x+y+z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3y^2z + 3xy^2 + 3xz^2 + 3yz^2 + 6xyz$

Theorem 5.5.1 (Multinomial Theorem)

$$(x_1 + x_2 + x_3 + \cdots + x_k)^n = \sum \binom{n}{n_1\ n_2 \ldots n_k} x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_k^{n_k}$$

where the sum includes a term for each non-negative integer solution of $n_1 + n_2 + \cdots + n_k = n$.

Example What is the coefficient of $x^5 y z^3$ in $(w+x+y+z)^9$?

Solution: This term is $w^0 x^5 y^1 z^3$. By Multinomial Theorem its coefficient is $\binom{9}{0\ 5\ 1\ 3} = \frac{9!}{0!\,5!\,1!\,3!}$

$$= \frac{9\cdot 8\cdot 7\cdot 6\cdot \cancel{5!}}{3!\ \cancel{5!}} = \frac{9\cdot 8\cdot 7\cdot 6}{6} = 9\cdot 8\cdot 7 = \boxed{504}$$

Thus the term in question is $504x^5yz^3$

Question How many terms does $(w+x+y+z)^9$ have?

Solution By the Multinomial theorem, the sum has a term for each non-negative solution of $n_1+n_2+n_3+n_4=9$. We know from previous techniques that the number of solutions is

$$\binom{9+4-1}{9} = \binom{12}{9} = \frac{12\cdot 11\cdot 10\cdot 9!}{9!\,(12-9)!} = \frac{12\cdot 11\cdot 10}{3!} = 220 \text{ terms.}$$

Obviously we don't want to write them all out, but the important thing is that the multinomial theorem allows us to find any one of them that we want.

Next we'll explore how Pascal's formula generalizes to the multinomial coefficients.

First note that the binomial coefficients are special cases of multinomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k\ \ n-k}$$

Note $k+(n-k)=n$

Moreover Pascals formula looks like this:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n}{k\ \ n-k} = \binom{n-1}{k-1\ \ n-k} + \binom{n-1}{k\ \ n-1-k}$$

$$\boxed{\binom{n}{a\ \ b} = \binom{n-1}{a-1\ \ b} + \binom{n-1}{a\ \ b-1}}$$

Formula

$$\boxed{\binom{n}{n_1\ n_2\ n_3\ \cdots\ n_k} = \binom{n-1}{(n_1-1)\ n_2\ n_3\ \cdots\ n_k} + \binom{n-1}{n_1\ (n_2-1)\ n_3\ \cdots\ n_k} + \binom{n-1}{n_1\ n_2\ (n_3-1)\ \cdots\ n_k} + \cdots + \binom{n-1}{n_1\ n_2\ n_3\ \cdots\ (n_k-1)}}$$

Read the proof in the book, but you can get an idea of why it works by considering the following special case.

Consider finding the coefficient of $x^a y^b z^c$ in:

$$(x+y+z)^n$$

$$= (x+y+z)(x+y+z)^{n-1}$$

$$= (x+y+z)\sum \binom{n-1}{l\ \ p\ \ q} x^l y^p z^q$$

$$= (x+y+z)\left(\cdots + \binom{n-1}{a-1\ b\ c} x^{a-1} y^b z^c + \cdots + \binom{n-1}{a\ b-1\ c} x^a y^{b-1} z^c + \cdots + \binom{n-1}{a\ b\ c-1} x^a y^b z^{c-1} + \cdots\right)$$

<u>Note</u> Coefficient of $x^a y^b z^c$ is $\binom{n}{a\ b\ c}$ which by above is

$$\binom{n}{a\ b\ c} = \binom{n-1}{a-1\ \ b\ \ c} + \binom{n-1}{a\ \ b-1\ \ c} + \binom{n-1}{a\ \ b\ \ c-1}$$

<u>Section 5.6</u> <u>Newton's Binomial Theorem.</u>

<u>Goal:</u> Generalize Binomial Theorem so it works for $(x+y)^\alpha$ with $\alpha \in \mathbb{R}$

First, note $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$ ← Makes sense when $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}^+$

<u>Ex</u> $\binom{4}{6} = \frac{4(4-1)(4-2)(4-3)(4-4)(4-5)}{6!} = 0$

<u>Ex</u> $\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6} = \frac{\frac{3}{8}}{6} = \frac{1}{16}$

Also recall that the MacLaurin series for a function $f(x)$ is

$$f(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

and this series converges to $f(x)$ on some radius of convergence around $x = 0$.

Now lets make the MacLauren series for $(1+x)^\alpha$

$$(1+x)^\alpha = \frac{(1+0)^\alpha}{0!}x^0 + \frac{\alpha(1+0)^{\alpha-1}}{1!}x^1 + \frac{\alpha(\alpha-1)(1+0)^{\alpha-2}}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)(1+0)^{\alpha-3}}{3!}x^3 + \cdots$$

$$= 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4$$

$$= \binom{\alpha}{0}x^0 + \binom{\alpha}{1}x^1 + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \binom{\alpha}{4}\alpha^4 + \cdots$$

Newtons Binomial Theorem:

$$(1+x)^\alpha = \sum_{K=1}^{\infty}\binom{\alpha}{k}x^k \quad (\text{interval of convergence is } |x|<1)$$

Note:

$$\left(1+\frac{y}{x}\right)^\alpha = \sum_{K=0}^{\infty}\binom{\alpha}{K}\left(\frac{y}{x}\right)^k$$

$$x^\alpha\left(1+\frac{y}{x}\right)^\alpha = x^\alpha\sum_{K=0}^{\infty}\binom{\alpha}{K}y^k x^{-k}$$

$$(x+y)^\alpha = \sum_{K=0}^{\infty}\binom{\alpha}{K}x^{\alpha-k}y^k$$

Alternate form of Newton's Binomial Theorem

$$(x+y)^\alpha = \sum_{K=0}^{\infty}\binom{\alpha}{k}x^{\alpha-k}y^k$$

Ex $\sqrt{2} = (1+1)^{\frac{1}{2}} \approx \binom{\frac{1}{2}}{0}1^0 + \binom{\frac{1}{2}}{1}1^1 + \binom{\frac{1}{2}}{2}1^2 + \binom{\frac{1}{2}}{3}1^3 + \binom{\frac{1}{2}}{4}1^4$

$$= 1 + \frac{\frac{1}{2}}{1!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}$$

$$= 1 + \frac{1}{2} - \frac{1}{8} + \frac{\frac{3}{8}}{3!} - \frac{\frac{15}{16}}{4!} = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} = 1.3928$$