Section 3.5 Combinations of Multisets

Definition An r-combination of a multiset S is an unordered selection of r objects from S.

Example $S = \{a, a, a, b, c, c\}$

0-combinations of S: $\emptyset$

1-combinations of S: $\{a\}$, $\{b\}$, $\{c\}$

2-combinations of S: $\{a,a\}$, $\{a,b\}$ $\{a,c\}$ $\{b,c\}$, $\{c,c\}$

3-combinations of S $\{a,a,a\}$, $\{a,a,b\}$ $\{a,a,c\}$, $\{a,b,c\}$, $\{a,c,c\}$ $\{b,c,c\}$

4-combinations of S $\{a,a,a,b\}$ $\{a,a,a,c\}$ $\{a,a,b,c\}$, $\{a,a,c,c\}$, $\{a,b,c,c\}$

5-combinations of S $\{a,a,a,b,c\}$ $\{a,a,a,c,c\}$ $\{a,a,b,c,c\}$

6-combinations of S $\{a,a,a,b,c,c\}$

7-combinations of S NONE

General Question: Given a multiset S, and integer r, how many different r-combinations of S?

In general, the answer to this question relies on techniques that we have not yet developed. However there is an easy answer in the case that each element of S has a repetition number that is equal to r, or greater.

Example $S = \{\underset{\infty}{a}, \underset{\infty}{b}, \underset{\infty}{c}\}$

How many 6-combinations? e.g.

$\{aa, bbb, c\}$ aa|bbb|c

$\{aaaab, c\}$ aaaa|b|c

$\{a, b, cccc\}$ a|b|cccc

$\{b, ccccc\}$ |b|ccccc

$\{bbbbbb\}$ |bbbbbb|

Any 6-combination can be encoded as $\underbrace{**\cdots*}_{\text{* for each a}} \mid \underbrace{**\cdots*}_{\text{* for each b}} \mid \underbrace{**\cdots*}_{\text{* for each c}}$

Such a configuration is a list of length $6+2=8$, with 2 spots selected for the bars, and the remaining spots filled with stars. Thus $\binom{8}{2}$ different 3-combinations

Answer: $\binom{6+2}{2} = \binom{8}{2} = \frac{8 \cdot 7}{2} = 28$ 6-combinations

Example $S = \{\underset{\infty}{a}, \underset{\infty}{b}, \underset{\infty}{c}, \underset{\infty}{d}\}$

How many different 10-combinations?

$\overbrace{\underbrace{**\cdots*}_{a\text{'s}} \mid \underbrace{**\cdots*}_{b\text{'s}} \mid \underbrace{**\cdots*}_{c\text{'s}} \mid \underbrace{**\cdots*}_{d\text{'s}}}^{\text{fill 3 out of 10+3 slots with |'s}}$

Answer: $\binom{10+3}{3} = \binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2} = 13 \cdot 2 \cdot 11 = \boxed{153}$ 10-combos

General problem:

Given a multiset $S = \{ {}_r a_1, {}_r a_2 \cdots {}_r a_k \}$ where each element has repitition number r (or greater), how many different r-combinations does it have?

Such an r-combination has structure

$$\{ a_1, a_1, \cdots a_1, \quad a_2, a_2 \cdots a_2, \quad a_3, a_3, \cdots a_3 \cdots \quad a_k, a_k, \cdots a_k \}$$

$$* * \cdots * \,|\, * * \cdots * \,|\, * * \cdots * \,|\, \cdots \,|\, * * \cdots *$$

This is a list of r stars and $k-1$ separating bars. Such a list has length $r+k-1$. You can make it by starting with $r+k-1$ empty spots, selecting $k-1$ of them for the bars and filling the remaining spots with stars.

Thus the number of r-combinations is $\binom{r+k-1}{k-1}$

Alternatively you could start with $r+k-1$ empty spots, choose r of them for stars, and fill the remaining $k-1$ spots with bars. In this way there are $\binom{r+k-1}{r}$ different r-combinations.

Conclusion

Theorem 3.5.1 Suppose $S = \{ {}_{n_1} a_1 \ {}_{n_2} a_2 \cdots {}_{n_k} a_k \}$ is a multiset containing k different objects $a_1\ a_2 \cdots a_k$, where each a_i has repetition number $n_i > r$. (for example $n_i = \infty$) Then the total number of r-combinations of S is

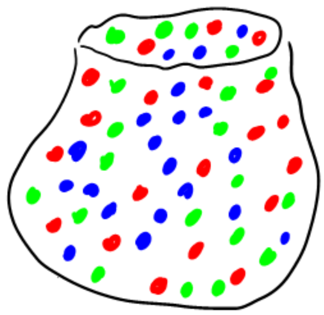
$$\binom{r+k-1}{k-1} = \binom{r+k-1}{r}$$

Notation Although Brualdi does not do it, some authors use the notation

$$\left(\!\binom{k}{r}\!\right) = \binom{r+k-1}{r} = \left(\begin{matrix}\text{number of } r\text{-combinations of a multiset } S \\ \text{of } k \text{ objects, each with repetition number } \infty\end{matrix}\right)$$

Example A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 20 of them. How many different outcomes?

e.g. 6 red, 4 green 10 blue
or 19 red 0 green 1 blue, etc.



Solution: Your grab is 20-combination of the multiset $S = \{ {}_{20}\text{red}, {}_{20}\text{green}, {}_{20}\text{blue} \}$

number of such outcomes is $\left(\!\binom{3}{20}\!\right) = \binom{20+3-1}{20} = \binom{22}{2} =$

$= \frac{22 \cdot 21}{2} = 231.$

Alternative Approach You grab 20 marbles.

Let x = number of red marbles drawn
y = number of green marbles drawn
z = number of blue marbles drawn

Then $x+y+z = 20$ and $x, y, z \geq 0$.
Thus the answer to our question is the number of non-negative integer solutions of $x+y+z=20$.

We saw how to solve such a question in a previous lecture.

$$\underbrace{**\,*\cdots*}_{x} \,\Big|\, \underbrace{**\cdots*}_{y} \,\Big|\, \underbrace{**\cdots*}_{z}$$

Any solution is encoded as a list of length $20+2 = 22$ where we select 20 spots for stars and fill the remaing two spots with bars. The number of solutions is then

$$\binom{22}{20} = \frac{22!}{20!\,(22-2)!} = \frac{20 \cdot 21}{2} = 231.$$