Section 3.2 Permutations of Sets

Recall that a list of length n is an ordered sequence $(a_1\ a_2\ a_3 \cdots a_n)$ of n elements.

Examples

$()$ ← length 0 (empty list)
(A) ← length 1
(A, B) ← length 2
(A, B, C) ← length 3
(H, T, H, T, T) ← length 4

Common abreviations

$(A, B, C) = ABC$
$(A, C, B) = ABC$
$(H, T, H, H) = HTHH$

However, notice there is no such abreviations of the empty list. We always write it as $()$

Our first topic today is to review the idea of the factorial of a number. It is an operation that counts the number of length-n lists made from n symbols, with no repetition

n	Set S with n elements	lists made from elements of S, no repetition	Number of such lists	
0	$\{\} = \phi$	$()$	1	$= 0!$
1	$\{A\}$	A	1	$= 1!$
2	$\{A, B\}$	AB BA	2	$= 2!$
3	$\{A, B, C\}$	ABC ACB BAC BCA CAB CBA	6	$= 3!$
4	$\{A, B, C, D\}$	ABCD ABDC ACBD ACDB ADBC ADCB BACD BADC BCAD BCDA BDAC BDCA CABD CADB CBAD CBDA CDAB CDBA DABC DACB DBAC DBCA DCAB DCBA	24	$= 4!$

Definition The factorial of a positive integer n, denoted $n!$ is the number of non-repetitive lists made from n elements, with no repetition

From above, $0! = 1$ $\quad 1! = 1$ $\quad 2! = 2$ $\quad 3! = 6$ $\quad 4! = 24$.

By the multiplication principle we have the following formula

Formula If $n \geq 1$ then $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

Examples: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ $\qquad 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ etc.

<u>Definition</u> A <u>permutation</u> of a set $S = \{a_1, a_2, \cdots, a_n\}$ is an arrangement of the elements of S into a list of length n.

<u>Example</u> $S = \{A, B, C\}$
Permutations of S: ABC ACB BAC BCA CAB CBA

Thus if S has n elements there are $n!$ permutations of S.

<u>Definition</u> If $|S| = n$ and $1 \le r \le n$, an <u>r-permutation</u> of S is an arrangement of r elements of S in a list

<u>Example</u> $S = \{A, B, C, D\}$

1-permutations of S: A B C D
2-permutations of S: AB BA AC CA AD DA
BC CB BD DB CD DC
3-permutations of S: ABC ACB ABD ADB ACD ADC BCD etc
4-permutations of S: ABCD BACD ... etc. ($4! = 24$ of these)

\# of 2-permutations of S is $\underline{4} \cdot \underline{3} = 12$
\# of 3-permutations of S is $\underline{4} \cdot \underline{3} \cdot \underline{2} = 24$
} using multiplication principle

Reasoning as above, if $|S| = n$, then \# of r-permutations of S is

$$\underset{\uparrow \atop 1^{st}}{\underline{n}}\ \underset{\uparrow \atop 2^{nd}}{\underline{(n-1)}}\ \underset{\uparrow \atop 3^{rd}}{\underline{(n-2)}}\ \underset{\uparrow \atop 4^{th}}{\underline{(n-3)}} \cdots \underset{\uparrow \atop n^{th}}{\underline{n-r+1}} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-r)!}$$

Here is a summary:

<u>Notation and Formula</u> The number of r-permutations of an n-element set is $P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$ This formula assumes $0 \le r \le n$. If $r < 0$ or $r > n$ then $P(n,r) = 0$

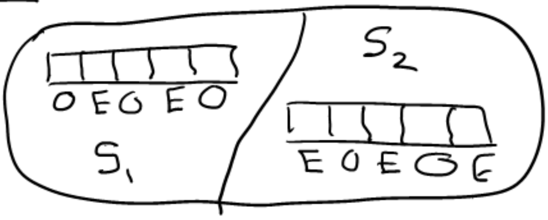
<u>Example</u> Number of 5-letter words made from A, B, C, D, E, F, G, H, I (without repetition) is

$$P(9,5) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$$

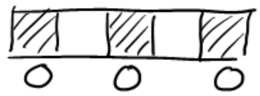
$$\text{or } \frac{9!}{(9-5)!} = \frac{9!}{4!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!} = 9\cdot 8\cdot 7\cdot 6\cdot 5 = 15120$$

Example In how many ways can we arrange five of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 in a row in such a way that the digits alternate between even and odd?

(Example 07812 or 12345, but not 30214)



Solution Partition set S of such lists into two parts S_1 and S_2, like this →
Then the answer will be $|S| = |S_1| + |S_2|$ (by the addition principle) so we just need to find $|S_1|$ and $|S_2|$

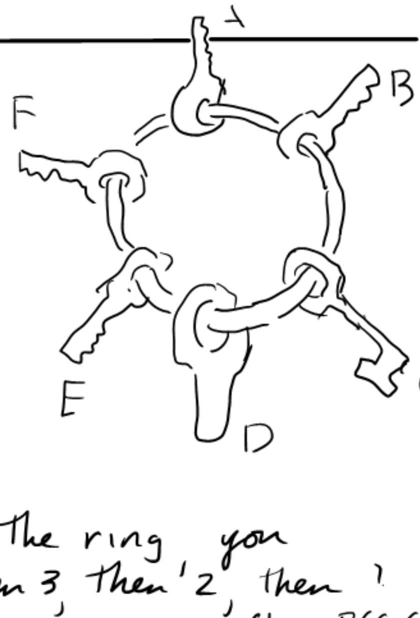


To find $|S_1|$, first fill in the odd digits. Here we are arranging 4 odd digits in 3 positions, so there are $P(4,3)$ ways to do this. Next we arrange 2 of the 5 even digits into the two remaining slots. There are $P(5,2)$ ways to do this. Then by the multiplication principle
$|S_1| = P(4,3)\,P(5,2) = (4\cdot 3\cdot 2)(5\cdot 4) = 24\cdot 20 = 480$

Next let's find $|S_2|$. Here we start by arranging 2 of the 4 odd digits in their two slots. There are $P(4,2)$ ways to do this
Next we have to arrange 3 of the 5 even digits into 3 slots, and there are $P(5,3)$ ways to do this. By the multiplication principle, $|S_2| = P(4,2)\cdot P(5,3) = (4\cdot 3)(5\cdot 4\cdot 3) = 12\cdot 60 = 720$

Answer $|S| = P(4,3)P(5,2) + P(4,2)P(5,3) = 480 + 720 = \boxed{1200}$
Thus there are 1200 such arrangements.

Circular Permutations.



So far our permutations have arranged things in a line, but you can also arrange them in a circle. This leads to the idea of circular permutations.

As motivation, consider the question of how many ways we can arrange six keys on a key ring
Starting at A and moving clockwise around the ring, you have 5 choices for the next key, then 4, then 3, then 2, then 1.
Thus the total # of arrangements is $5\cdot 4\cdot 3\cdot 2\cdot 1 = 120 = \frac{6!}{6} = \frac{P(6,6)}{6}$

Formula (Read in text)
The number of circular r-permutations of n things is $\frac{P(n,r)}{r}$

§3.4 Permutations of Multisets

Definition A *multiset* is a set in which elements are allowed to be repeated. The number of times an element is repeated is called its *multiplicity* or *repitition number*.

Ex Multiset: $S = \{a, a, a, b, b, c\} = \{3\cdot a, 2\cdot b, 1\cdot c\} = \{\underset{3}{a}, \underset{2}{b}, \underset{1}{c}\}$

3-permutations of S: aaa, aab, aac, abb, abc, bbc
aba, aca, bab, acb, bcb
baa, caa, bba, bac, cbb
bca
cab
cba

4-permutations of S: aaab aaac aabb aabc abbc etc.

6-permutations of S: – just called *permutations* of S as $|S| = 6$
aaabbc abcbaa, etc.

Ex Multiset $\{a, a, a, \ldots, b, b, b, b, \ldots, c, c, c, \ldots\} = \{\underset{\infty}{a}, \underset{\infty}{b}, \underset{\infty}{c}\}$

3-permutations of S

aaa; aab aba baa; aac aca caa; abb bab bba; acc cac cca; abc acb bca bac cab cba; bbb; bbc bcb cbb; bcc cbc ccb; ccc

Question: How many r-permutations of a multiset S?

Theorem 3.4.1 The multiset $S = \{\underset{\infty}{a_1}, \underset{\infty}{a_2}, \underset{\infty}{a_3}, \ldots, \underset{\infty}{a_k}\}$ has k^r r-permutations

Example $S = \{\underset{\infty}{a}, \underset{\infty}{b}, \underset{\infty}{c}\}$ has $3^3 = 27$ 3-permutations (see above).
and $3^4 = 81$ 4-permutations

Now let's consider multisets whose elements have finite multiplicity.

Example How many anagrams of "LOCK"
i.e. how many 4-permutations of $S = \{\underset{1}{L}, \underset{1}{O}, \underset{1}{C}, \underset{1}{K}\}$?
Ans $P(4,4) = 24$

Example How many anagrams of "LOOK"
i.e. how many 4-permutations of $S = \{\underset{1}{L}, \underset{2}{O}, \underset{1}{K}\}$

To get an idea of how to proceed, make one of the O's lower-case. The anagrams are

LoOK LOoK	KoOL KOoL	OoKL oOKL	OoLK oOLK
LOKo LoKO	KOLo KoLO	oKOL OKoL	oLOK OLoK
LKOo LKoO	KLOo KLoO	oKLO OKLo	oLKO OLKo

$P(4,4) = 24$ 4-permutations

For each one the O's can be arranged in $2! = 2$ ways, resulting in same anagram.

Ans $\dfrac{4!}{2!} = 12$ anagrams of LOOK

Ex How many anagrams of PEPPERMINT

i.e. How many 10-combinations of $S = \left\{ \begin{array}{ccccccc} P, & E, & R, & M, & I, & N, & T \\ 3 & 2 & 1 & 1 & 1 & 1 & 1 \end{array} \right\}$?

To reason this out, suppose the 3 P's and 2 E's are distinct. So we are counting anagrams of PEppeRMINT

PEppeRMINT $3! \cdot 2!$ of these	PINTMEppeR $3! \cdot 2!$ of these	etc.

← All 10! anagrams

Ans Total number of anagrams of PEPPERMINT is $\dfrac{10!}{3!\ 2!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2} = 302400$

Reasoning in this way we get the following:

Theorem 3.4.2 Given a multiset $S = \left\{ \begin{array}{ccccc} a_1 & a_2 & a_3 & \dots & a_k \\ n_1 & n_2 & n_3 & \dots & n_k \end{array} \right\}$

the number of permutations of S is $\dfrac{k!}{n_1!\, n_2! \dots n_k!}$