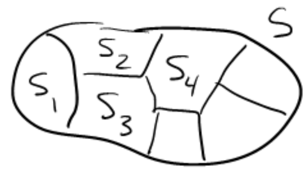
Chapter 3 Permutations and Combinations

Section 3.1 Two Basic Counting Principles

Today we take up two very important counting principles that will be with us for the entire course: The <u>addition principle</u> and the <u>multiplication principle</u>. They are easy to understand, but the real challenge is recognizing how and when they can be applied. Creativity is essential.

<u>Recall</u> A <u>partition</u> of a set S is a collection of non-empty subsets $S_1, S_2, \ldots, S_n \subseteq S$ for which $S = S_1 \cup S_2 \cup \cdots \cup S_n$ and $S_i \cap S_j = \phi$ for each index $i \neq j$. The subsets $S_1, S_2, \ldots, S_n$ are called the <u>parts</u> of the partition



<u>Addition Principle</u>:
Suppose a set S is partitioned into parts as $S = S_1 \cup S_2 \cup \cdots \cup S_n$. Then $\lvert S \rvert = \lvert S_1 \rvert + \lvert S_2 \rvert + \cdots + \lvert S_n \rvert$.

Addition principle is used when we need to count the elements of S (i.e. find $\lvert S \rvert$) and can break this task into the more managable tasks of computing each $\lvert S_i \rvert$

Before looking at examples, let's get to the multiplication principle. It involves the idea of <u>a list</u>. A <u>list</u> of length n is an ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$

Examples $(5, 7, 7, 3, 5) \xrightarrow{\text{or}} 5\ 7\ 7\ 3\ 5$

 $\xrightarrow{\text{or}}$ 

$(H, H, H, T, H, T, T) \xrightarrow{\text{or}} H\,H\,H\,T\,H\,T\,T$

In a list, entries may be repeated, and changing the order results in a different list

$$804\ 355\ 3963 \neq 804\ 355\ 9363$$

Also lists of different lengths are unequal:

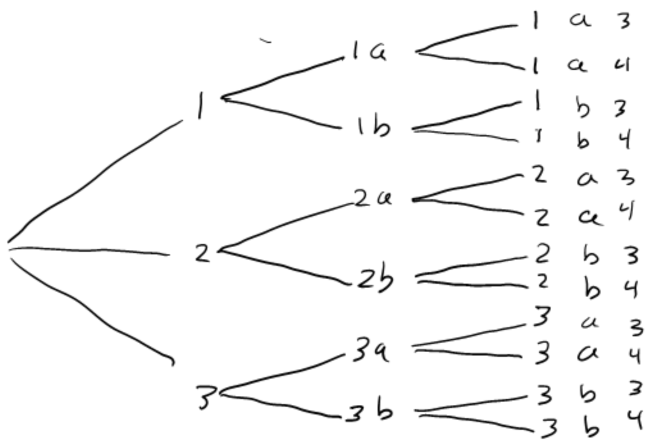
$$00000 \neq 0000$$

The multiplication principle is a means of counting lists. To motivate it consider the following question:

A list is to be made as follows $\begin{cases} 1^{st} \text{ entry is from } \{1,2,3\} \\ 2^{nd} \text{ entry is from } \{a,b\} \\ 3^{rd} \text{ entry is from } \{3,4\} \end{cases}$

How many such lists are possible?

We can enumerate these lists with a tree showing 3 branches for the first entry, 2 for the second and 2 for the third. This enumerates all possible lists on the right, and we can then see our answer



In total there are $3 \cdot 2 \cdot 2 = 12$ such possible lists. This leads to the multiplication principle.

Multiplication Principle

Suppose that in making a list of length n we have
P_1 choices for the 1^{st} entry
P_2 choices for the 2^{nd} entry
$\vdots$
P_n choices for the n^{th} entry
Then the total number of possible lists is $P_1 P_2 P_3 \cdots P_n$.

Alternate formulation.

Suppose that in performing an n-step task we have
P_1 choices for 1^{st} step
P_2 choices for 2^{nd} step
$\vdots$
P_n choices for n^{th} step
Then the total number of ways to do the task is $P_1 P_2 P_3 \cdots P_n$

We will now look at examples that apply our new principles. One issue that arizes here — and that will be with us for the entire course — is that there are two types of lists:

Repitition Allowed e.g. phone numbers 804-355-7727, etc.

Repetition Not Allowed e.g. permutations of $\{a,b,c\}$
abc, acb, bac, bca, cab, cba,

Examples: A list of length 4 is to be made with symbols A B C D E F G. How many such lists are there if

(a) Repetition is allowed
(b) Repetition not allowed
(c) Repetition not allowed and list must contain an E
(d) Repetition allowed and list must contain an E

(a) Solution: List formed like this

7 choices	7 choices	7 choices	7 choices

Therefore there are $7 \cdot 7 \cdot 7 \cdot 7 = 2401$ such lists.

(b) Solution: List formed like this

7 choices	6 choices	5 choices	4 choices

Therefore there are $7 \cdot 6 \cdot 5 \cdot 4 = 840$ such lists.

(c) Solution: Here an immediate application of the multiplication principle is problematic. If we (say) chose E for the first entry then we are locked in to that choice for the rest of the problem, and we will miss those lists that have the E in a later position. Instead we will combine the addition and multiplication principles.

Divide the possible lists into 4 types according to whether the E appears in the first, second, third or fourth positions

Count lists of type S_1 as:

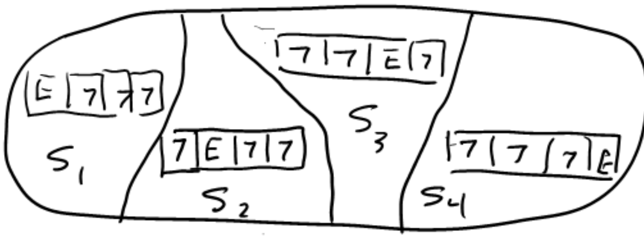
E	6 choices	5 choices	4 choices

So $|S_1| = 6 \cdot 5 \cdot 4 = 120$. Similarly $|S_2| = |S_3| = |S_4| = 120$ where we have used the multiplication principle for each S_i. Now the addition principle gives the total number of lists as $|S_1| + |S_2| + |S_3| + |S_4| = 120 + 120 + 120 + 120 = \underline{480}$

(d) Solution $= \begin{pmatrix}\text{number of lists with}\\ \text{repetition allowed}\\ \text{from ABCDEFG}\end{pmatrix} - \begin{pmatrix}\text{number of lists with}\\ \text{repetition allowed}\\ \underline{\text{not}}\text{ containing E}\end{pmatrix}$

$= 7 \cdot 7 \cdot 7 \cdot 7 - 6 \cdot 6 \cdot 6 \cdot 6 = 2401 - 1296 = \underline{1105}$

Be careful here. What if we tried the approach from (c) for (d)? The (incorrect) answer would be



$|S_1| + |S_2| + |S_3| + |S_4| =$

$7 \cdot 7 \cdot 7 + 7 \cdot 7 \cdot 7 + 7 \cdot 7 \cdot 7 + 7 \cdot 7 \cdot 7 = 1372$

Here we _overcounted_ because, for example the list EEAB is included in both S_1 and S_2. In combinatorics we must always be careful not to double count in this way.