MATH 525

Combinatorics

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The prequisite for this course is MATH 300 or its equivalent. You are expected to have a firm grip on the material from MATH 300, Including: sets, set operations, logic, direct proof, contrapositive proof, contradiction proof, induction, if-and-only-if proof, and functions. All of this forms the foundation of combinatorics.

Combinatorics is the mathematics of counting and enumerating. Thus it deals largely with finite or discrete mathematical structures.

In the first week we will examine three basic Counting principles that will be with us for the entire semester:

- Pigeonhole Principle
- Addition Principle
- Multiplication Principle

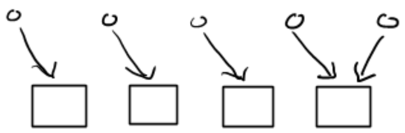
We will begin with the pigeonhole principle. In essense, it says that if more than n pigeons fly into fewer than n pigeonholes, then at least one pigeonhole contains more than one pigeon.

Section 2.1 The Pigeonhole Principle

Here is a useful idea that we will use occasionally in this course. It's obvious, but can be used to prove many non-obvious things

Pigeonhole Principle

If more than n objects are placed in n boxes, then at least one box contains more than one object

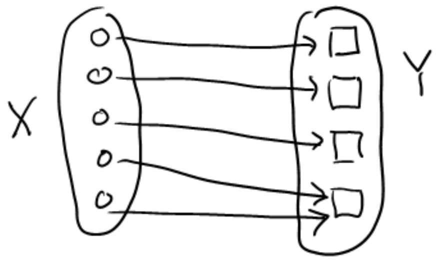


5 objects

4 boxes

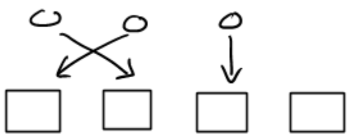
Alternate Formulation

Given function $f: X \to Y$, if $|X| > |Y|$ then f is not injective



Related useful facts

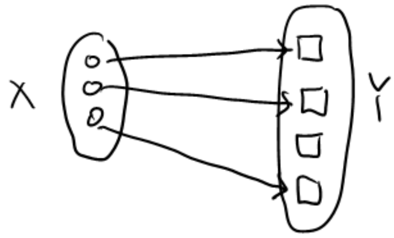
If fewer than n objects are placed in n boxes, then one box remains empty



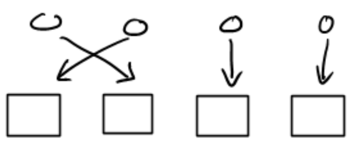
3 objects

4 boxes

Given $f: X \to Y$, if $|X| < |Y|$ then f is not surjective



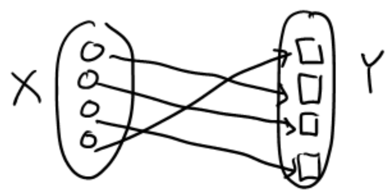
If n objects are placed in n boxes and no box gets more than one object, then each box gets an object



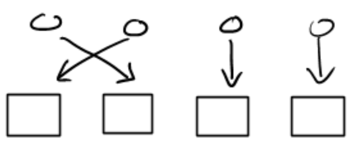
4 objects

4 boxes

Given $f: X \to Y$, if $|X| = |Y|$ and f is injective, then f is also surjective



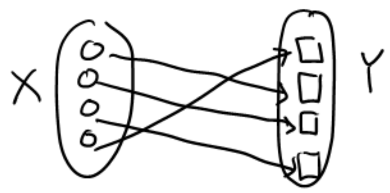
If n objects are placed in n boxes and no box is empty, then each box contains exactly one object



4 objects

4 boxes

Given $f: X \to Y$ if $|X| = |Y|$ and f is surjective, then f is also injective



Example Six numbers between 0 and 9 are chosen.
Show that two of them sum to 9.

6 Objects: numbers a_1 a_2 a_3 a_4 a_5 a_6

5 Boxes: labeled as:

0, 9	1, 8	2, 7	3, 6	4, 5

Rule Put a_i in box with a_i in its label.

By PHP some box contains more than one number.
Those two numbers a_i and a_j sum to 9, i.e. $a_i + a_j = 9$

Formal Proof
For any six numbers between 0 and 9, two of them sum to 9.

Proof Let $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be six such numbers
Put $Y = \{\{0,9\}, \{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}\} = \{\{n, 9-n\} \mid 0 \le n \le 4\}$
Note $|X| = 6 > 5 = |Y|$
Define $f: X \to Y$ as $f(a_i) = \{n, 9-n\}$ with $a_i \in \{n, 9-n\}$
Because $|X| > |Y|$, PHP says f not injective.
Thus there is $a_i \neq a_j$ with $f(a_i) = f(a_j) = \{n, 9-n\}$
Then $a_i = n$, $a_j = 9-n$ OR $a_i = 9-n$, $a_j = n$. Either way $a_i + a_j = 9$

Proposition Six integers are chosen at random. Show that the sum or difference of two of them is a multiple of 9.

Proof Choose six integers $a_1\ a_2\ a_3\ a_4\ a_5\ a_6$. Put $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$
By division algorithm, $a_i = 9q_i + r_i$ where $0 \le r_i < 9$ is remainder when a_i is divided by 9.
Let $Y = \{r_i \mid 1 \le i \le 6\}$ Note $|X| \ge |Y|$
Define $f: X \to Y$ by $f(a_i) = r_i$ = remainder in $9\overline{)a_i}$
CASE I $|X| = 6 = |Y|$. Then $Y = \{r_1\ r_2\ r_3\ r_4\ r_5\ r_6\}$
By previous result $r_i + r_j = 9$ for some i & j.
Then $a_i + a_j = (9q_i + r_i) + (9q_j + r_j) = 9q_i + 9q_j + 9 = 9(q_i + q_j + 1)$
CASE II $|X| > |Y|$. By PHP, $f: X \to Y$ not injective, so $f(a_i) = r_i = f(a_j)$. Now $a_i - a_j = (9q_i + r_i) - (9q_j + r_i) = 9(q_i - q_j)$.

Section 2.2 Pigeonhole Principle - strong Form

Use for Exercise 2.4.14

Strong Form of Pigeonhole Principle.

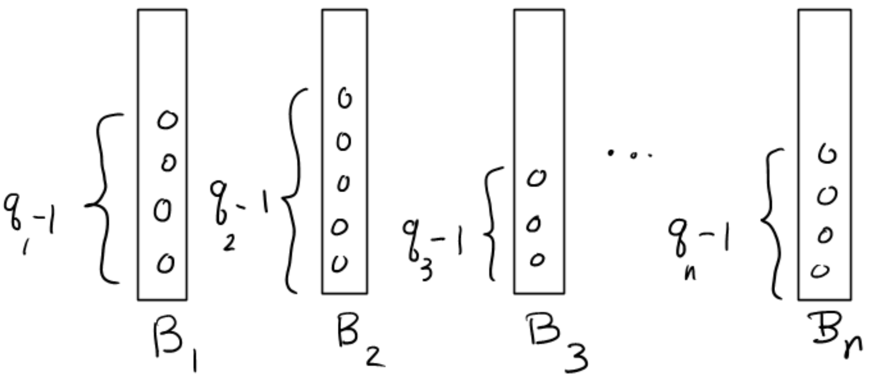
Suppose you have n boxes $\boxed{B_1}$ $\boxed{B_2}$ $\boxed{B_3}$ $\cdots$ $\boxed{B_n}$.

and positive integers $q_1 \quad q_2 \quad q_3 \quad \cdots \quad q_n$

If you place $q_1 + q_2 + \cdots + q_n - n + 1$ objects in the n boxes then some box B_i contains at least q_i elements.

<u>Why it works</u> Suppose you had that many objects but tried to place <u>fewer</u> than q_i objects each box B_i.

Like this



So far you've put a total of $\sum (q_i - 1)$ $= q_1 + q_2 + \cdots q_n - n$ objects in the boxes. There is one more object that needs to be placed. It will give $q_i - 1 + 1 = q_i$ objects in some box B_i.

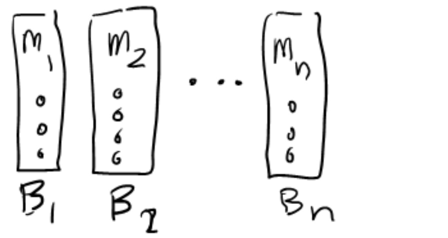
The strong form has a useful cousin that you have used often

> If the average of non-negative integers $m_1 \, m_2 \cdots m_n$ is greater than $r-1$, then at least one of the integers is greater than or equal to r.

To see that this is really an instance of the strong form of the PHP, suppose $\frac{m_1 + m_2 + \cdots m_n}{n} > r-1$,

so $\sum m_i > nr - n$, i.e $\sum m_i \geq nr - n + 1$.

Now put m_i objects in box B_i



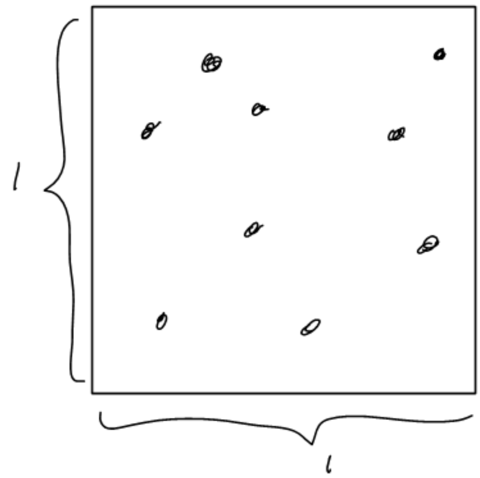
Set $q_1 = r \quad q_2 = r \quad \cdots \quad q = r$

Number of objects in n boxes is $\sum m_i \geq nr - n + 1 = q_1 + q_2 + \cdots + q_n - n + 1$

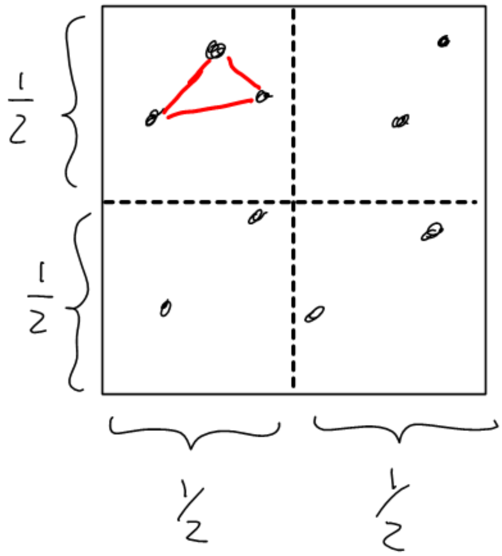
PHP says some box B_i contains r or more objects.

That is, $m_i \geq r$.

Example Suppose 9 points are placed inside a 1x1 square. Show that 3 of the points form a triangle whose area is $\frac{1}{8}$ square unit or less.



Solution: Divide the square into 4 smaller squares of side-length $1/2$. These are our "boxes."



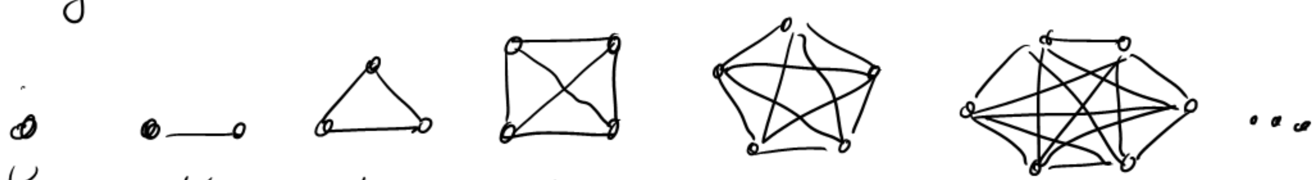
We have now placed 9 objects in 4 boxes. The average number of objects per box is

$$\frac{9}{4} > 2$$

Therefore some box contains 3 points (objects). These 3 points form a triangle inside of a $\frac{1}{2} \times \frac{1}{2}$ square. The area of the $\triangle$ must be at most half the area of the $\square$, i.e. at at most $\frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{8}$ square units.

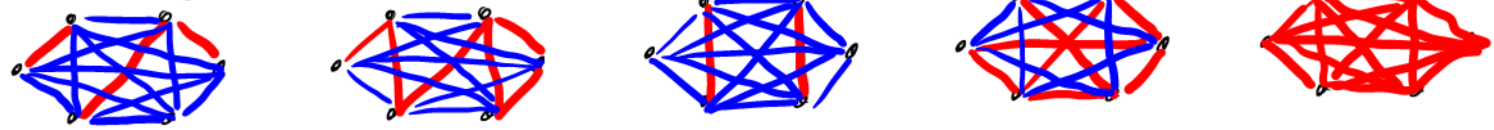
Section 2.3 A Theorem of Ramsey

First, a definition. The symbol K_n stands for the configuration of n points with line segments joining any two of them.



K_1 K_2 K_3 K_4 K_5 K_6 ...

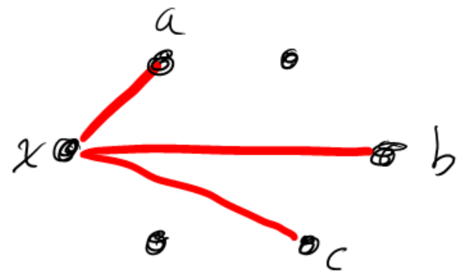
Investigation



Color the edges of K_6 arbirarily red or blue. No matter how you do it, there is always a red or blue K_3.

In fact this is always true, no matter how you color the K_6

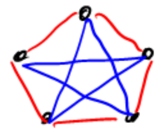
To see why, take an arbitrary coloring. At any vertex x, there are 5 edges with endpoint x. By the pigeonhole principle, at least 3 of these have the same color. (say red).



Now if any of the edges ab, bc, ca are colored red, we get a red triangle K_3 (one of whose vertices is x). Otherwise these edges are all blue, so we get a blue K_3 abc.

We express this as $K_6 \to K_3\, K_3$ meaning no matter how we color the edges of K_6 red or blue, we always get either a red K_3 or a blue K_3 somewhere in the K_6.

But notice that $K_5 \not\to K_3, K_3$ because of this coloring:



In general, $K_p \to K_m\, K_n$ means that no matter how we color the edges ok K_p red or blue, it must contain a red K_m or a blue K_n

Also $K_p \not\to K_m\, K_n$ means its possible to color the edges of K_p so that there is neither a monochromatic K_m nor a K_n

Ramseys Theorem (Proof difficult)
For any positive integers m, n, there is a p for which $K_p \to K_m\, K_n$

Definition: The Ramsey Number $r(m,n)$ is the least p for which $K_p \to K_m\, K_n$

Example $r(3,3) = 6$ (by above).

Known $r(3,4) = 9$ $r(3,5) = 14$ $r(3,6) = 18$ $r(3,7) = 23$

However most Ramsey numbers are unknown: $40 \le r(3,10) \le 43$

Read text: It explains how this generalizes the pigeonhole principle