§7.4, 7.5 More on Generating Functions.

Recall the following formulas

Binomial Theorem

Newton's Binomial Theorem

- $\dfrac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n$
- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$
- $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$
- $(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots + \binom{\alpha}{k}x^k + \cdots$
- $(1-x)^\alpha = 1 - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \binom{\alpha}{3}x^3 + \cdots + (-1)^k\binom{\alpha}{k}x^k + \cdots$
- $\dfrac{1}{(1-rx)^n} = 1 + \binom{n}{1}rx + \binom{n+1}{2}r^2x^2 + \binom{n+2}{3}r^3x^3 + \cdots + \binom{n+k-1}{k}r^kx^k + \cdots$

Today we will need a formula for $\sqrt{1-rx} = (1-rx)^{\frac{1}{2}} = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-r)^k x^k$

For this, note that for $k>0$

$$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{3-2k}{2}\right)}{k!}$$

$$= \frac{(-1)^{k-1}}{2^k}\,\frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-3)}{k!} = \frac{(-1)^{k-1}}{2^k}\,\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-3)(2k-2)}{2\cdot 4\cdot 6\cdots(2k-2)\;k!}$$

$$= \frac{(-1)^{k-1}}{2^k 2^{k-1} k}\,\frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{(-1)^{k-1}}{k\,2^{2k-1}}\binom{2k-2}{k-1}$$

In summary $\boxed{\binom{\frac{1}{2}}{0} = 1}$ and for $k>0$ $\boxed{\binom{1/2}{k} = \frac{(-1)^{k-1}}{k\,2^{2k-1}}\binom{2k-2}{k-1}}$

Now $\sqrt{1-rx} = \binom{\frac{1}{2}}{0}r^0x^0 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\,2^{2k-1}}\binom{1/2}{k}(-r)^kx^k = 1 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\,2^{2k-1}}\binom{2k-2}{k-1}(-r)^kx^k$

$$= 1 + \sum_{k=1}^{\infty}\frac{(-1)^{2k-1}}{k\,2^{2k-1}}\binom{2k-2}{k-1}r^kx^k = 1 - \sum_{k=1}^{\infty}\frac{1}{k\,2^{2k-1}}\binom{2k-2}{k-1}r^kx^k$$

Therefore

$$\boxed{\bullet\ \sqrt{1-rx} = 1 - \sum_{k=1}^{\infty}\frac{1}{k\,2^{2k-1}}\binom{2k-2}{k-1}r^kx^k}$$

Ex In how many ways can we pick 25 balls from an unlimited supply of balls of 7 different colors, with between 2 and 6 of each color?

Solution Let h_n be the number of ways to pick n balls. We seek h_{25}

Generating function is $g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots$

$$= (x^2 + x^3 + x^4 + x^5 + x^6)^7 = x^{14}(1 + x + x^2 + x^3 + x^4)^7$$

$$= x^{14}\left(\frac{1 - x^5}{1 - x}\right)^7 = x^{14}(1 - x^5)^7 \frac{1}{(1-x)^7}$$

$$= x^{14}\left(1 - \binom{7}{1}x^5 + \binom{7}{2}x^{10} + \binom{7}{3}x^{15} + \binom{7}{4}x^{20} + \binom{7}{5}x^{25} + \binom{7}{6}x^{30} + \binom{7}{7}x^{35}\right)$$

$$\cdot \left(1 + \binom{7}{1}x + \binom{8}{2}x^2 + \binom{9}{3}x^3 + \binom{10}{4}x^4 + \binom{11}{5}x^5 + \binom{12}{6}x^6 + \cdots + \binom{17}{11}x^{11} + \cdots\right)$$

$$= x^{14}\left(\cdots + \left[\binom{17}{11} - \binom{7}{1}\binom{12}{6} + \binom{7}{2}\binom{7}{1}\right]x^{11} + \cdots\right)$$

$$= \cdots + \left(\binom{17}{11} - 7\binom{12}{6} + \binom{7}{2}\cdot 7\right)x^{25} + \cdots$$

Answer $\binom{17}{11} - 7\binom{12}{6} + 7\binom{7}{2}$

$$= \frac{17\cdot 16\cdot \overset{3}{\cancel{15}}\cdot \overset{7}{14}\cdot 13\cdot \cancel{12}}{6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} - 7\frac{\cancel{12}\cdot 11\cdot \overset{2}{\cancel{10}}\cdot \overset{3}{9}\cdot \overset{2}{\cancel{8}}\cdot 7}{\cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} + 7\,\frac{7\cdot 6}{2}$$

$$= 17\cdot 8\cdot 7\cdot 13 - 7\cdot 11\cdot 2\cdot 3\cdot 2\cdot 7 + 7\cdot 7\cdot 3$$

$$= \boxed{6055}$$

§7.6 A Geometry Example

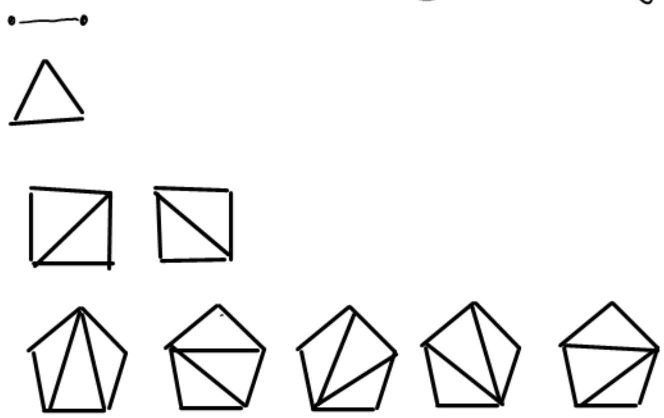
This section is concerned with the following example.
In how many ways can a convex $(n+1)$-gon be divided into triangles by connecting diagonals? Let h_n be the number of ways.
Strategy: Build a recurrence relation for h_n, then find all h_n by means of a generating function.

$n = 1$ $\qquad h_1 = 1$ (no diagonals)

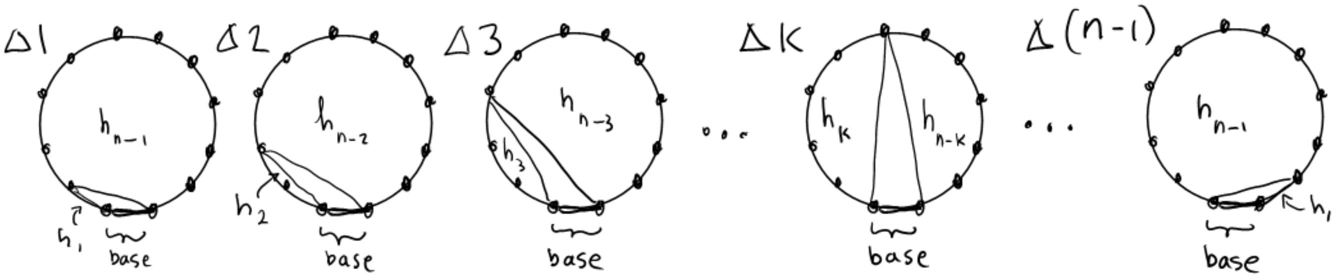
$n = 2$ $\qquad h_2 = 1$ (no diagonals)

$n = 3$ $\qquad h_3 = 2$

$n = 4$ $\qquad h_4 = 5$



n



For h_n pick an edge of the $(n+1)$-gon as a "base".
There are $n-1$ different triangles that could have this base. Call them $\Delta 1, \Delta 2, \Delta 3, \ldots, \Delta k, \ldots \Delta n-1$ as shown above.
The k^{th} triangle divides the $(n+1)$-gon into a triangle, a $(k+1)$-gon and an $(n-k+1)$-gon. The $(k+1)$-gon can be further divided in h_k ways and the $(n-k+1)$-gon can be divided into h_{n-k} ways. Thus the total number of divisions using Δk is $h_k h_{n-k}$.

Conclusion

The number of ways an $(n+1)$-gon can be divided into Δ's is

$$h_n = h_1 h_{n-1} + h_2 h_{n-2} + h_3 h_{n-3} + \ldots h_{n-1} h_1$$

with initial values 1, 1, 2, 5.

Note This recurrence relation is not linear and has no fixed degree.

Now we will solve the recurrence relation.
Let its generating function be

$$g(x) = h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + \cdots$$

Then $g(x)g(x) =$

$$= (h_1 x + h_2 x^2 + h_3 x^3 + \cdots)(h_1 x + h_2 x^2 + h_3 x^3 + \cdots)$$

$$= h_1^2 x^2 + (h_1 h_2 + h_2 h_1) x^3 + (h_1 h_3 + h_2 h_2 + h_3 h_1) x^4 + (h_1 h_4 + h_2 h_3 + h_3 h_2 + h_4 h_1) x^4$$

$$= h_2 x^2 + h_3 x^3 + h_4 x^4 + h_5 x^5 + \cdots$$

$$= g(x) - h_1 x = g(x) - x$$

Therefore $(g(x))^2 - g(x) + x = 0$

So $$g(x) = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot x}}{2 \cdot 1} = \frac{1}{2}\left(1 \pm \sqrt{1-4x}\right)$$

But as $g(0) = 0$ we must have $\boxed{g(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}}$

Then $g(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \sum_{k=1}^{\infty} \frac{1}{k\,2^{2k-1}} \binom{2k-2}{k-1} 4^k x^k\right)$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{2}{k} \binom{2k-2}{k-1} x^k = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n}\binom{2n-2}{n-1}}_{h_n} x^n$$

Solution $\boxed{h_n = \frac{1}{n}\binom{2n-2}{n-1}}$

$$h_4 = \frac{1}{4}\binom{8-2}{4-1} = \frac{1}{4}\binom{6}{3} = \frac{1}{4}\,\frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = \boxed{5}$$

$$h_5 = \frac{1}{5}\binom{10-2}{5-1} = \frac{1}{5}\binom{8}{4} = \frac{1}{5}\,\frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 7 \cdot 2 = \boxed{14}$$

These numbers h_n are called Catalan numbers, and we will study their properties in Chapter 8.