quadrants (each quadrant is determined by prescribing signs to each of x_1, x_2, x_3). More generally, n-dimensional space is coordinatized by associating an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ with each point. There are n coordinate planes, namely, those determined by $x_1 = 0$, $x_2 = 0, \ldots$, and $x_n = 0$. These planes divide n-dimensional space into the 2^n "quadrants" determined by prescribing signs to each of $x_1, x_2, \ldots, x_n$. One such quadrant is the so-called non-negative quadrant $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.

8.5 Exercises

1. Let $2n$ (equally spaced) points be chosen on a circle. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.

2. Prove that the number of 2-by-n arrays

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{bmatrix}$$

that can be made from the numbers $1, 2 \ldots, 2n$ so that

$$x_{11} < x_{12} < \cdots < x_{1n},$$

$$x_{21} < x_{22} < \cdots < x_{2n}$$

and

$$x_{11} < x_{21}, x_{12} < x_{22}, \ldots, x_{1n} < x_{2n},$$

equals the nth Catalan number, C_n.

3. Write out all the multiplication schemes for four numbers and the triangularization of a convex polygonal region of five sides corresponding to them.

4. Determine the triangularization of a convex polygonal region corresponding to the following multiplication schemes:

   (a) $(a_1 \times (((a_2 \times a_3) \times (a_4 \times a_5)) \times a_6))$

   (b) $(((a_1 \times a_2) \times (a_3 \times (a_4 \times a_5)) \times ((a_6 \times a_7) \times a_8)))$

5. $*$ Let m and n be non-negative integers with $n \geq m$. There are $m+n$ people in line to get into a theatre for which admisssion is 50 cents. Of the $m+n$ people, n have a 50 cents piece and m have a 1 dollar bill. The box office opens with an empty cash register. Show that the number of ways the people can line up so that change is available when needed is

$$\frac{n-m+1}{n+1}\binom{m+n}{m}.$$

(The case $m = n$ is the case treated in section 8.1.)

6. Let the sequence $h_0, h_1, \ldots, h_n, \ldots$ be defined by $h_n = 2n^2 - n + 3$, $(n \geq 0)$. Determine the difference table, and find a formula for $\sum_{k=0}^{n} h_k$.

7. The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries of the 0th row of its difference table are $1, -1, 3, 10$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.

8. Find the sum of the fifth powers of the first n positive integers.

9. Prove the following formula for the kth-order differences of a sequence $h_0, h_1, \ldots, h_n, \ldots$:

$$\Delta^k h_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} h_{n+j}.$$

10. If h_n is a polynomial in n of degree m, prove that the constants $c_0, c_1, \ldots, c_m$ such that

$$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_m\binom{n}{m}$$

are uniquely determined (cf. Theorem 8.2.2).

11. Compute the Stirling numbers of the second kind $S(8, k)$, $(k = 0, 1, \ldots, 8)$.

12. Prove that the Stirling numbers of the second kind satisfy the relations

    (a) $S(n, 1) = 1, \quad (n \geq 1)$

(b) $S(n, 2) = 2^{n-1} - 1, \quad (n \geq 2)$

(c) $S(n, n-1) = \binom{n}{2}, \quad (n \geq 1)$

(d) $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$

13. Let X be an p-element set and let Y be a k-element set. Prove that the number of functions $f : X \to Y$ which map X *onto* Y equals

$$k!S(p, k) = S^{\#}(p, k).$$

14. $*$ Find and verify a general formula for

$$\sum_{k=0}^{n} k^p$$

involving Stirling numbers of the second kind.

15. The number of partitions of a set of n elements into k distinguishable boxes (some of which may be empty) is k^n. By counting in a different way prove that

$$k^n = \binom{k}{1}1!S(n,1) + \binom{k}{2}2!S(n,2) + \cdots + \binom{k}{n}k!S(n,n).$$

(If $k > n$, define $S(n, k)$ to be 0.)

16. Compute the Bell number B_8 (cf. Exercise 11).

17. Compute the triangle of Stirling numbers of the first kind $s(n, k)$ up to $n = 7$.

18. Write $[n]_k$ as a polynomial in n for $k = 1, 2, \ldots, 7$.

19. Prove that the Stirling numbers of the first kind satisfy

(a) $s(n, 1) = (n-1)!, \quad (n \geq 1)$

(b) $s(n, n-1) = \binom{n}{2}, \quad (n \geq 1)$

20. Verify that $[n]_n = n!$, and write $n!$ as a polynomial in n using the Stirling numbers of the first kind. Do this explicitly for $n = 6$.

21. For each integer $n = 1, 2, 3, 4, 5$, construct the diagram of the set $\mathcal{P}_n$ of partitions of n partially ordered by majorization.

22. (a) Calculate $p(6)$ and construct the diagram of the set $\mathcal{P}_6$ partially ordered by majorization.

(b) Calculate $p(7)$ and construct the diagram of the set $\mathcal{P}_7$ partially ordered by majorization.

23. A total order on a finite set has a unique maximal element (a largest element) and a unique minimal element (a smallest element). What are the largest partition and smallest partition in the lexicographic order on $\mathcal{P}(n)$?

24. A partial order on a finite set may have many maximal elements and minimal elements. In the set $\mathcal{P}_n$ of partitions of n partially ordered by majorization, prove that there is a unique maximal element and a unique minimal element.

25. Let $t_1, t_2, \ldots, t_m$ be distinct positive integers, and let $q_n = q_n(t_1, t_2, \ldots, t_n)$ equal the number of partitions of n in which all parts are taken from $t_1, t_2, \ldots, t_m$. Define $q_0 = 1$. Show that the generating function for $q_0, q_1, \ldots, q_n, \ldots$ is

$$\prod_{k=1}^{m}(1 - x^{t_k})^{-1}.$$

26. Determine the conjugate of each of the following partitions:

(a) $12 = 5 + 4 + 2 + 1$

(b) $15 = 6 + 4 + 3 + 1 + 1$

(c) $20 = 6 + 6 + 4 + 4$

(d) $21 = 6 + 5 + 4 + 3 + 2 + 1$

(e) $30 = 8 + 6 + 6 + 4 + 3 + 2$

27. For each integer $n > 2$, determine a self-conjugate partition of n that has at least two parts.

28. Prove that conjugation reverses the order of majorization; that is, if λ and μ are partitions of n and λ is majorized by μ, then μ^* is majorized by λ^*.

29. Evaluate $h_{k-1}^{(k)}$, the number of regions into which k-dimensional space is partitioned by $k-1$ hyperplanes in general position.