Example. Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors, red, white, and blue, if an even number of squares are to be colored red.

Let h_n denote the number of such colorings where we define h_0 to be 1. Then h_n equals the number of n-permutations of a multiset of three colors (red, white, and blue), each with an infinite repetition number, in which red occurs an even number of times. Thus the exponential generating function for $h_0, h_1, \ldots, h_n, \ldots$ is the product of red, white, and blue factors:

$$\begin{aligned}
g^{(e)} &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\\
&= \frac{1}{2}(e^x+e^{-x})e^x e^x = \frac{1}{2}(e^{3x}+e^x)\\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} 3^n\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}(3^n+1)\frac{x^n}{n!}.
\end{aligned}$$

Hence $h_n = (3^n+1)/2$. □

Example. Determine the number h_n of n digit numbers with each digit odd where the digits 1 and 3 occur an even number of times.

Let $h_0 = 1$. The number h_n equals the number of n-permutations of the multiset $S = \{\infty\cdot 1, \infty\cdot 3, \infty\cdot 5, \infty\cdot 7, \infty\cdot 9\}$, in which 1 and 3 occur an even number of times. The exponential generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$ is a product of five factors, one for each of the allowable digits:

$$\begin{aligned}
g^{(e)}(x) &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2\left(1+x+\frac{x^2}{2!}+\cdots\right)^3\\
&= \left(\frac{e^x+e^{-x}}{2}\right)^2 e^{3x}\\
&= \left(\frac{e^{2x}+1}{2}\right)^2 e^x\\
&= \frac{1}{4}(e^{4x}+2e^{2x}+1)e^x\\
&= \frac{1}{4}(e^{5x}+2e^{3x}+e^x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\left(\sum_{n=0}^{\infty}5^n\frac{x^n}{n!}+2\sum_{n=0}^{\infty}3^n\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{5^n+2\times 3^n+1}{4}\right)\frac{x^n}{n!}.
\end{aligned}$$

Hence

$$h_n = \frac{5^n+2\times 3^n+1}{4}, \qquad (n\geq 0).$$

□

Example. Determine the number h_n of ways to color the squares of a 1-by-n board with the colors red, white, and blue where the number of red squares is even and there is at least one blue square.

The exponential generating function $g^{(e)}(x)$ is

$$\begin{aligned}
g^{(e)}(x) &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\\
&= \frac{e^x+e^{-x}}{2}e^x(e^x-1)\\
&= \frac{e^{3x}-e^{2x}+e^x-1}{2}\\
&= -\frac{1}{2}+\sum_{n=0}^{\infty}\frac{3^n-2^n+1}{2}\frac{x^n}{n!}.
\end{aligned}$$

Thus

$$h_n = \frac{3^n-2^n+1}{2}, \qquad (n = 1, 2, \ldots)$$

and

$$h_0 = 0.$$

Note that h_0 should be 0. A 1-by-0 board is empty, no squares get colored, and so we cannot satisfy the condition that the number of blue squares is at least 1. □

7.8 Exercises

1. Let $f_0, f_1, f_2, \ldots, f_n, \ldots$ denote the Fibonacci sequence. By evaluating each of the following expressions for small values of n, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence.

(a) $f_1 + f_3 + \cdots + f_{2n-1}$

(b) $f_0 + f_2 + \cdots + f_{2n}$

(c) $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n$

(d) $f_0^2 + f_1^2 + \cdots + f_n^2$

2. Prove that the nth Fibonacci number f_n is the integer which is closest to the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

3. Prove the following about the Fibonacci numbers:

(a) f_n is even if and only if n is divisible by 3.

(b) f_n is divisible by 3 if and only if n is divisible by 4.

(c) f_n is divisible by 4 if and only if n is divisible by 6.

(d) f_n is divisible by 5 if and only if n is divisible by 5.

(e) By examining the Fibonacci sequence, make a conjecture about when f_n is divisible by 7 and then prove your conjecture.

4. * Let m and n be positive integers. Prove that if m is divisible by n, then f_m is divisible by f_n.

5. * Let m and n be positive integers whose greatest common divisor is d. Prove that the greatest common divisor of the Fibonacci numbers f_m and f_n is the Fibonacci number f_d.

6. Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the two colors red and blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n.

7. Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

8. Suppose that in his problem Fibonacci had placed two pairs of rabbits in the enclosure at the beginning of a year. Find the number of pairs of rabbits in the enclosure after one year. More generally, find the number of pairs of rabbits in the enclosure after n months.

9. Solve the recurrence relation $h_n = 4h_{n-2}$, $(n \geq 2)$ with initial values $h_0 = 0$ and $h_1 = 1$.

10. Solve the recurrence relation $h_n = (n+2)h_{n-1}$, $(n \geq 1)$ with initial value $h_0 = 2$.

11. Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$ with initial values $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$.

12. Solve the recurrence relation $h_n = 8h_{n-1} - 16h_{n-2}$, $(n \geq 2)$ with initial values $h_0 = -1$ and $h_1 = 0$.

13. Solve the recurrence relation $h_n = 3h_{n-2} - 2h_{n-3}$, $(n \geq 3)$ with initial values $h_0 = 1$, $h_1 = 0$, and $h_2 = 0$.

14. Solve the recurrence relation $h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$ with initial values $h_0 = 0$, $h_1 = 1$, $h_2 = 1$, and $h_3 = 2$.

15. Solve the following recurrence relations by examining the first few values for a formula and then proving your conjectured formula by induction.

(a) $h_n = 3h_{n-1}$, $(n \geq 1)$; $h_0 = 1$

(b) $h_n = h_{n-1} - n + 3$, $(n \geq 1)$; $h_0 = 2$

(c) $h_n = -h_{n-1} + 1$, $(n \geq 1)$; $h_0 = 0$

(d) $h_n = -h_{n-1} + 2$, $(n \geq 1)$; $h_0 = 1$

(e) $h_n = 2h_{n-1} + 1$, $(n \geq 1)$; $h_0 = 1$

16. Let h_n denote the number of ways to perfectly cover a 1-by-n board with monominoes and dominoes in such a way that no two dominoes are consecutive. Find, but do not solve, a recurrence relation and initial conditions satisfied by h_n.

17. * Let $2n$ equally spaced points be chosen on a circle. Let h_n denote the number of ways to join these points in pairs so

that the resulting line segments do not intersect. Establish a recurrence relation for h_n.

18. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 4h_{n-1} + 3 \times 2^n, \quad (n \geq 1) \\ h_0 &= 1. \end{aligned}$$

19. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 3h_{n-1} - 2, \quad (n \geq 1) \\ h_0 &= 1. \end{aligned}$$

20. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 2h_{n-1} + n, \quad (n \geq 1) \\ h_0 &= 1. \end{aligned}$$

21. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 6h_{n-1} - 9h_{n-2} + 2n, \quad (n \geq 2) \\ h_0 &= 1 \\ h_1 &= 0. \end{aligned}$$

22. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 4h_{n-1} - 4h_{n-2} + 3n + 1, \quad (n \geq 2) \\ h_0 &= 1 \\ h_1 &= 2. \end{aligned}$$

23. Determine the generating function for each of the following sequences.

(a) $c^0 = 1, c, c^2, \ldots, c^n, \ldots$

(b) $1, -1, 1, -1, \ldots, (-1)^n, \ldots$

(c) $\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, \ldots, (-1)^n \binom{\alpha}{n}, \ldots,$

(α is a real number.)

(d) $1, \frac{1}{1!}, \frac{1}{2!}, \ldots, \frac{1}{n!}, \ldots$

(e) $1, -\frac{1}{1!}, \frac{1}{2!}, \ldots, (-1)^n \frac{1}{n!}, \ldots$

24. Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$. Determine the generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ where h_n is the number of n-combinations of S with the added restriction:

(a) Each e_i occurs an odd number of times.

(b) Each e_i occurs a multiple-of-3 number of times.

(c) The element e_1 does not occur, and e_2 occurs at most once.

(d) The element e_1 occurs $1, 3$, or 11 times, and the element e_2 occurs $2, 4$, or 5 times.

(e) Each e_i occurs at least 10 times.

25. Solve the following recurrence relations by using the method of generating functions as described in section 7.5.

(a) $h_n = 4h_{n-2}$, $(n \geq 2)$; $h_0 = 0, h_1 = 1$

(b) $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$; $h_0 = 1, h_1 = 3$

(c) $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$; $h_0 = 0, h_1 = 1, h_2 = 2$

(d) $h_n = 8h_{n-1} - 16h_{n-2}$, $(n \geq 2)$; $h_0 = -1, h_1 = 0$

(e) $h_n = 3h_{n-2} - 2h_{n-3}$, $(n \geq 3)$; $h_0 = 1, h_1 = 0, h_2 = 0$

(f) $h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$; $h_0 = 0, h_1 = 1, h_2 = 1, h_3 = 2$

26. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 4h_{n-1} + 4^n, \quad (n \geq 1) \\ h_0 &= 3. \end{aligned}$$

27. Determine the generating function for the sequence of cubes

$$0, 1, 8, \ldots, n^3, \ldots.$$

28. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by

$$h_n = n^3, \quad (n \geq 0).$$

Show that $h_n = h_{n-1} + 3n^2 - 3n + 1$ is the recurrence relation for the sequence.

29. Formulate a combinatorial problem that leads to the following generating generating function:

$$(1+x+x^2)(1+x^2+x^4+x^6)(1+x^2+x^4+\cdots)(x+x^2+x^3+\cdots).$$

30. Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which there are an even number of apples, at most two oranges, a multiple of three number of bananas, and at most one pear. Then find a formula for h_n from the generating function.

31. Determine the generating function for the number h_n of non-negative integral solutions of

$$2e_1 + 5e_2 + e_3 + 7e_4 = n.$$

32. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by $h_n = \binom{n}{2}$, $(n \geq 0)$. Determine the generating function for the sequence.

33. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by $h_n = \binom{n}{3}$, $(n \geq 0)$. Determine the generating function for the sequence.

34. $*$ Let h_n denote the number of regions into which a convex polygonal region with $n+2$ sides is divided by its diagonals, assuming no three diagonals have a common point. Define $h_0 = 0$. Show that

$$h_n = h_{n-1} + \binom{n+1}{3} + n, \quad (n \geq 1).$$

Then determine the generating function and from it obtain a formula for h_n.

35. Determine the exponential generating function for the sequence of factorials: $0!, 1!, 2!, 3!, \ldots, n!, \ldots$.

36. Let α be a real number. Let the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ be defined by $h_0 = 1$, and $h_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$, $(n \geq 1)$. Determine the exponential generating function for the sequence.

37. Let S denote the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \ldots, \infty \cdot e_k\}$. Determine the exponential generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ where $h_0 = 1$ and for $n \geq 1$:

(a) h_n equals the number of n-permutations of S in which each object occurs an odd number of times.

(b) h_n equals the number of n-permutations of S in which each object occurs at least four times.

(c) h_n equals the number of n-permutations of S in which e_1 occurs at least once, e_2 occurs at least twice, $\ldots$, e_k occurs at least k times.

(d) h_n equals the number of n-permutations of S in which e_1 occurs at most once, e_2 occurs at most twice, $\ldots$, e_k occurs at most k times.

38. Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even, and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, \ldots, h_n, \ldots$, and then find a simple formula for h_n.

39. Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors red, blue, green, and orange if an even number of squares are to be colored red and an even number are to be colored green.

40. Determine the number of n digit numbers with all digits odd, such that 1 and 3 each occur a non-zero, even number of times.

41. Determine the number of n digit numbers with all digits at least 4, such that 4 and 6 each occur an even number of times, and 5 and 7 each occur at least once, there being no restriction on the digits 8 and 9.