while if $n = 4$ they are

4 1 3 2	4 3 2 1	4 2 1 3
3 2 1 4	3 2 4 1	2 1 4 3
2 4 3 1	2 4 1 3	3 1 4 2
1 3 2 4	1 4 3 2	

Thus $Q_1 = 1$, $Q_2 = 1$, $Q_3 = 3$, and $Q_4 = 11$.

Theorem 6.5.1 *For* $n \geq 1$

$$Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!.$$

Proof. Let S be the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. Let P_j be the property that in a permutation the pattern $j(j+1)$ does occur, $(j = 1, 2, \ldots, n-1)$. Thus a permutation of $\{1, 2, \ldots, n\}$ is counted in the number Q_n if and only if it has none of the properties $P_1, P_2, \ldots, P_{n-1}$. As usual let A_j denote the set of permutations of $\{1, 2, \ldots, n\}$ which satisfy property P_j, $(j = 1, 2, \ldots, n-1)$. Then

$$Q_n = |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_{n-1}|,$$

and we apply the inclusion-exclusion principle to evaluate Q_n. We first calculate the number of permutations in A_1. A permutation is in A_1 if and only if the pattern 12 occurs in it. Thus a permutation in A_1 may be regarded as a permutation of the $n-1$ symbols $\{12, 3, 4, \ldots, n\}$. We conclude that $|A_1| = (n-1)!$, and in general we see that

$$|A_j| = (n-1)! \qquad (j = 1, 2, \ldots, n-1).$$

Permutations which are in two of the sets $A_1, A_2, \ldots, A_{n-1}$ contain two patterns. These patterns either share an element, like the patterns 12 and 23 or have no element in common, like the patterns 12 and 34 . A permutation which contains the two patterns 12 and 34 can be regarded as a permutation of the $n-2$ symbols $\{12, 34, 5, \ldots, n\}$. Thus $|A_1 \cap A_3| = (n-2)!$. A permutation which contains the two patterns 12 and 23 contains the pattern 123 and thus can be regarded as a permutation of the $n-2$ symbols $\{123, 4, \ldots, n\}$. Thus $|A_1 \cap A_2| = (n-2)!$. In general, we see that

$$|A_i \cap A_j| = (n-2)!$$

for each 2-combination $\{i, j\}$ of $\{1, 2, \ldots, n-1\}$. More generally, we see that a permutation which contains k specified patterns from the list $12, 23, \ldots, (n-1)n$ can be regarded as a permutation of $n-k$ symbols, and thus that

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$$

for each k-combination $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n-1\}$. Since for each $k = 1, 2, \ldots, n-1$ there are $\binom{n-1}{k}$ k-combinations of $\{1, 2, \ldots, n-1\}$, applying the inclusion-exclusion principle we obtain the formula in the theorem. □

Using the formula of Theorem 6.5.1, we calculate that

$$Q_5 = 5! - \binom{4}{1}4! + \binom{4}{2}3! - \binom{4}{3}2! + \binom{4}{4}1! = 53.$$

The numbers $Q_1, Q_2, Q_3, \ldots$ are closely related to the derangement numbers. Indeed we have $Q_n = D_n + D_{n-1}$, $(n \geq 2)$ (see Exercise 23). Thus knowing the derangement numbers, we can calculate the numbers $Q_1, Q_2, Q_3, \ldots$. Since we have already seen in the preceding section that $D_5 = 44$, $D_6 = 265$, we conclude that $Q_6 = D_6 + D_5 = 265 + 44 = 309$.

6.6 Exercises

1. Find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 5, or 6.

2. Find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 6, 7, or 10.

3. Find the number of integers between 1 and 10,000 which are neither perfect squares nor perfect cubes.

4. Determine the number of 12-combinations of the multiset

$$S = \{4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}.$$

5. Determine the number of 10-combinations of the multiset

$$S = \{\infty \cdot a, 4 \cdot b, 5 \cdot c, 7 \cdot d\}.$$

6. A bakery sells chocolate, cinnamon, and plain doughnuts and at a particular time has 6 chocolate, 6 cinnamon, and 3 plain. If a box contains 12 doughnuts, how many different boxes of doughnuts are possible?

7. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in non-negative integers x_1, x_2, x_3, and x_4 not exceeding 8.

8. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in positive integers x_1, x_2, x_3, and x_4 not exceeding 8.

9. Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

which satisfy

$$1 \le x_1 \le 6,\ 0 \le x_2 \le 7,\ 4 \le x_3 \le 8,\ 2 \le x_4 \le 6.$$

10. Let S be a multiset with k distinct objects whose repetition numbers are $n_1, n_2, \ldots, n_k$, respectively. Let r be a positive integer such that there is at least one r-combination of S. Show that in applying the inclusion-exclusion principle to determine the number of r-combinations of S, one has $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$.

11. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

12. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which exactly four integers are in their natural position.

13. Determine the number of permutations of $\{1, 2, \ldots, 9\}$ in which at least one odd integer is in its natural position.

14. Determine a general formula for the number of permutations of the set $\{1, 2, \ldots, n\}$ in which exactly k integers are in their natural positions.

15. At a party 7 gentlemen check their hats. In how many ways can their hats be returned so that

(a) no gentleman receives his own hat?
(b) at least one of the gentlemen receives his own hat?
(c) at least two of the gentlemen receive their own hats?

16. Use combinatorial reasoning to derive the identity

$$\begin{aligned} n! &= \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} \\ &\quad + \cdots + \binom{n}{n-1}D_1 + \binom{n}{n}D_0. \end{aligned}$$

(Here D_0 is defined to be 1.)

17. Determine the number of permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 2 \cdot c\}$$

where, for each type of letter, the letters of the same type do not appear consecutively. (Thus *abbbbcaca* is not allowed, but *abbbacacb* is.)

18. Verify the factorial formula

$$n! = (n-1)((n-2)! + (n-1)!), \qquad (n = 2, 3, 4, \ldots).$$

19. Using the evaluation of the derangement numbers as given in Theorem 6.3.1, provide a proof of the relation

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \qquad (n = 3, 4, 5, \ldots).$$

20. Starting from the formula $D_n = nD_{n-1} + (-1)^n$, $(n = 2, 3, 4, \ldots)$, give a proof of Theorem 6.3.1.

21. Prove that D_n is an even number if and only if n is an odd number.

22. Show that the numbers Q_n of section 6.5 can be rewritten in the form

$$Q_n = (n-1)!\left(n - \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right).$$

23. (Continuation of Exercise 22.) Verify the identity

$$(-1)^k \frac{n-k}{k!} = (-1)^k \frac{n}{k!} + (-1)^{k-1} \frac{1}{(k-1)!},$$

and use it to prove that $Q_n = D_n + D_{n-1}$, $(n = 2, 3, \ldots)$.

24. What is the number of ways to place six non-attacking rooks on the 6-by-6 boards with forbidden positions as shown?

(a)

×	×				
		×	×		
				×	×

(b)

×	×				
×	×				
		×	×		
		×	×		
				×	×
				×	×

(c)

×	×				
	×	×			
		×			
				×	×
					×

25. Count the permutations $i_1 i_2 i_3 i_4 i_5 i_6$ of $\{1, 2, 3, 4, 5, 6\}$ where $i_1 \neq 1, 5$; $i_3 \neq 2, 3, 5$; $i_4 \neq 4$ and $i_6 \neq 5, 6$.

26. Count the permutations $i_1 i_2 i_3 i_4 i_5 i_6$ of $\{1, 2, 3, 4, 5, 6\}$ where $i_1 \neq 1, 2, 3$; $i_2 \neq 1$; $i_3 \neq 1$; $i_5 \neq 5, 6$ and $i_6 \neq 5, 6$.

27. Eight girls are seated around a carousel. In how many ways can they change seats so that each has a different girl in front of her?

28. Eight boys are seated around a carousel but facing inward, so that each boy faces another. In how many ways can they change seats so that each faces a different boy?

29. How many circular permutations are there of the multiset

$$\{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\}$$

where for each type of letter, all letters of that type do not appear consecutively?

30. How many circular permutations are there of the multiset

$$\{2 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}$$

where for each type of letter, all letters of that type do not appear consecutively?