1. $A^+ \neq A$ (and thus $|A^+| < |A|$) since there is a minimal element not in $A$;

2. $A^- \neq A$ (and thus $|A^-| < |A|$) since there is a maximal element not in $A$;

3. $A^+ \cap A^- = A$ since if there were an element $x$ in $A^+ \cap A^-$ not in $A$, then we would have $a_1 < x < a_2$ for some elements $a_1$ and $a_2$ in $A$, contradicting the assumption that $A$ is an antichain;

4. $A^+ \cup A^- = X$ since if there were an element $x$ not in $A^+ \cup A^-$, $A \cup \{x\}$ would be an antichain of larger size than $A$.

We apply the induction assumption to the smaller partially ordered sets $A^+$ and $A^-$ and conclude that $A^+$ can be partitioned into $m$ chains $E_1, E_2, \ldots, E_m$, and $A^-$ can be partitioned into $m$ chains $F_1, F_2, \ldots, F_m$. The elements of $A$ are the maximal elements of $A^-$ and so the last elements on the chains $F_1, F_2, \ldots, F_m$; the elements of $A$ are also the minimal elements of $A^+$ and so the first elements on the chains $E_1, E_2, \ldots, E_m$. We "glue" the chains together in pairs to form $m$ chains which partition $X$.

Case 2. There are at most two antichains of size $m$, one or both of the set of all maximal elements and the set of all minimal elements. Let $x$ be a minimal element and $y$ a maximal element with $x \leq y$ ($x$ may equal $y$). Then the largest size of an antichain of $X - \{x, y\}$ is m-1. By the induction hypothesis, $X - \{x, y\}$ can be partitioned into $m - 1$ chains. These chains together with the chain $x \leq y$ gives a partition of $X$ into $m$ chains. □

## 5.8 Exercises

1. Prove Pascal's formula by substituting the values of the binomial coefficients as given in equation (5.1).

2. Fill in the rows of Pascal's triangle corresponding to $n = 9$ and 10.

3. Consider the sum of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left. The first few are: $1, 1, 1+1 = 2, 1+2 = 3, 1+3+1 = 5, 1+4+3 = 8$. Compute several more of these diagonal sums, and determine how these sums are related. (Compare them with the values of the counting function $f$ in Exercise 4 of Chapter 1.)



4. Expand $(x+y)^5$ and $(x+y)^6$, using the binomial theorem.

5. Expand $(2x-y)^7$, using the binomial theorem.

6. What is the coefficient of $x^5y^{13}$ in the expansion of $(3x-2y)^{18}$? What is the coefficient of $x^8y^9$? (There is not a misprint in this last question!)

7. Use the binomial theorem to prove that
$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$
Generalize to find the sum
$$\sum_{k=0}^{n} \binom{n}{k} r^k$$
for any real number $r$.

8. Use the binomial theorem to prove that
$$2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$$

9. Evaluate the sum
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k.$$

10. Use combinatorial reasoning to prove the identity (5.2). (Hint: Think of choosing a team with one person designated as captain.)

11. Use *combinatorial* reasoning to prove the identity (in the form given)
$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$
(Hint: Let $S$ be a set with three distinguished elements $a, b$, and $c$ and count certain $k$-combinations of $S$.)

12. Let $n$ be a positive integer. Prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

13. Find one binomial coefficient equal to the following expression

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}.$$

14. Prove that

$$\binom{r}{k} = \frac{r}{r-k}\binom{r-1}{k}$$

for $r$ a real number and $k$ an integer with $r \neq k$.

15. Prove that for every integer $n > 1$

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0.$$

16. By integrating the binomial expansion, prove that for a positive integer $n$,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

17. Prove the identity in the previous exercise by using (5.2) and (5.3).

18. Evaluate the sum

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + (-1)^n\frac{1}{n+1}\binom{n}{n}.$$

19. Sum the series $1^2 + 2^2 + 3^2 + \cdots + n^2$ by observing that

$$m^2 = 2\binom{m}{2} + \binom{m}{1}$$

and using the identity (5.14).



20. Find integers $a, b$, and $c$ such that

$$m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}.$$

for all $m$. Then sum the series $1^3 + 2^3 + 3^3 + \cdots + n^3$.

21. Prove that for all real numbers $r$ and all integers $k$,

$$\binom{-r}{k} = (-1)^k\binom{r+k-1}{k}.$$

22. Prove that for all real numbers $r$ and all integers $k$ and $m$,

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

23. Every day a student walks from her home to school, which is located 10 blocks east and 14 blocks north from home. She always takes a shortest walk of 24 blocks.

    (a) How many different walks are possible?

    (b) Suppose that 4 blocks east and 5 blocks north of her home lives her best friend, whom she meets each day on her way to school. Now how many different walks are possible?

    (c) Suppose, in addition, that 3 blocks east and 6 blocks north of her friend's house there is a park where the two girls stop each day to rest and play. Now how many different walks are there?

    (d) Stopping at a park to rest and play, the two students often get to school late. To avoid the temptation of the park, our two students decide never to pass the intersection where the park is. Now how many different walks are there?

24. Consider a three-dimensional grid whose dimensions are 10 by 15 by 20. You are at the front lower left corner of the grid and wish to get to the back upper right corner 45 "blocks" away. How many different routes are there in which you walk exactly 45 blocks?

25. Use a combinatorial argument, to prove the *Vandermonde convolution* for the binomial coefficients: for all positive integers $m_1, m_2$, and $n$,

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$

Deduce the identity (5.11) as a special case.

26. Find and prove a formula for

$$\sum_{\substack{r,s,t \geq 0 \\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$$

where the summation extends over all nonnegative integers $r, s$ and $t$ with sum $r + s + t = n$.

27. Prove that the only clutter of $S = \{1, 2, 3, 4\}$ of size 6 is the clutter of all 2-combinations of $S$.

28. Prove that there are only two clutters of $S = \{1, 2, 3, 4, 5\}$ of size 10 (10 is maximum by Sperner's Theorem), namely, the clutter of all 2-combinations of $S$ and the clutter of all 3-combinations.

29. * Let $S$ be a set of $n$ elements. Prove that if $n$ is even, the only clutter of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the clutter of all $\frac{n}{2}$-combinations; if $n$ is odd, prove that the only clutters of this size are the clutter of all $\frac{n-1}{2}$-combinations and the clutter of all $\frac{n+1}{2}$-combinations.

30. Construct a partition of the combinations of $\{1, 2, 3, 4, 5\}$ into symmetric chains.

31. In a partition of the combinations of $\{1, 2, \ldots, n\}$ into symmetric chains, how many chains have only one combination in them? two combinations? $k$ combinations?

32. A talk show host has just bought 10 new jokes. Each night he tells some of the jokes. What is the largest number of nights on which you can tune in so that you never hear on one night at least all the jokes you heard on *one* of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1, 2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)



33. Prove the identity of Exercise 23, using the binomial theorem and the relation $(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m_1+m_2}$.

34. Use the multinomial theorem to show that for positive integers $n$ and $t$

$$t^n = \sum \binom{n}{n_1\ n_2 \cdots n_t}$$

where the summation extends over all non-negative integral solutions $n_1, n_2, \ldots, n_t$ of $n_1 + n_2 + \cdots + n_t = n$.

35. Use the multinomial theorem to expand $(x_1 + x_2 + x_3)^4$.

36. Determine the coefficient of $x_1^3 x_2 x_3^4 x_5^2$ in the expansion of

$$(x_1 + x_2 + x_3 + x_4 + x_5)^{10}.$$

37. What is the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expansion of

$$(x_1 - x_2 + 2x_3 - 2x_4)^9?$$

38. Expand $(x_1 + x_2 + x_3)^n$ by observing that

$$(x_1 + x_2 + x_3)^n = ((x_1 + x_2) + x_3)^n$$

and then using the binomial theorem.

39. Prove the identity (5.16) by a combinatorial argument. (Hint: Consider the permutations of a multiset of objects of $t$ different types with repetition numbers $n_1, n_2, \ldots, n_t$, respectively. Partition these permutations according to what type of object is in the first position.)

40. Prove by induction on $n$ that, for $n$ a positive integer,

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k, \qquad |z| < 1.$$

Assume the validity of

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \qquad |z| < 1.$$

41. Use Newton's binomial theorem to approximate $\sqrt{30}$.

42. Use Newton's binomial theorem to approximate $10^{1/3}$.

43. Use Theorem 5.7.1 to show that if $m$ and $n$ are positive integers, then a partially ordered set of $mn + 1$ elements has a chain of size $m + 1$ or an antichain of size $n + 1$.

44. Use the result of the previous exercise to show that a sequence of $mn + 1$ real numbers either contains an increasing subsequence of $m + 1$ numbers or a decreasing subsequence of $n + 1$ numbers (see Application 9 of section 2.2).

45. Consider the partially ordered set $(\{1, 2, \ldots, 12\}, |)$ of the first 12 positive integers partially ordered by "is divisible by."

    (a) Determine a chain of largest size and a partition of $\{1, 2, \ldots, 12\}$ into the smallest number of antichains.

    (b) Determine an antichain of largest size and a partition of $\{1, 2, \ldots, 12\}$ into the smallest number of chains.

# Chapter 6

# The Inclusion-Exclusion Principle and Applications

In this chapter we derive and apply an important counting formula called the inclusion-exclusion principle. Recall that the addition principle gives a simple formula for counting the number of objects in a union of sets, *provided the sets do not overlap*, that is, provided the sets determine a partition. The inclusion-exclusion principle gives a formula for the most general of circumstances where the sets are free to overlap without restriction. The formula is necessarily more complicated but, as a result, is more widely applicable.

## 6.1 The Inclusion-Exclusion Principle

In Chapter 3 we have seen several examples where it is easier to make an indirect count of the number of objects in a set rather than to count the objects directly. Two more examples are the following.

**Example.** Count the permutations $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ in which 1 is not in the first position (that is, $i_1 \neq 1$).

We could make a direct count by observing that the permutations with 1 not in the first position can be divided into $n - 1$ parts according to which of the $n - 1$ integers $k$ from $\{2, 3, \ldots, n\}$ is in the first position. A permutation with $k$ in the first position consists of $k$ followed by a permutation of the $(n - 1)$-element set $\{1, \ldots, k - 1, k + 1, \ldots, n\}$. Hence there are $(n - 1)!$ permutations of $\{1, 2, \ldots, n\}$ with $k$ in the first position. By the addition principle