□

Example. Continuing with the doughnut example following Theorem 3.5.1 we see in a similar way that the number of different boxes of doughnuts containing at least one doughnut of each of the 8 varieties equals

$$\binom{4+8-1}{4} = \binom{11}{4} = 330.$$

□

General lower bounds on the number of times each type of object occurs in the combination also can be handled by a change of variable.

Example. What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

in which

$$x_1 \geq 3,\ x_2 \geq 1,\ x_3 \geq 0 \text{ and } x_4 \geq 5?$$

We introduce new variables:

$$y_1 = x_1 - 3,\ y_2 = x_2 - 1,\ y_3 = x_3,\ y_4 = x_4 - 5,$$

and our equation becomes

$$y_1 + y_2 + y_3 + y_4 = 11.$$

The lower bounds on the x_i's are satisfied if and only if the y_i's are non-negative. The number of non-negative integral solutions of the new equation is

$$\binom{11+4-1}{11} = \binom{14}{11} = 364.$$

□

It is more difficult to count the number of r-combinations of a multiset

$$S = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$$

with k types of objects and general repetition numbers $n_1, n_2, \ldots, n_k$. The number of r-combinations of S is the same as the number of integral solutions of

$$x_1 + x_2 + \cdots + x_k = r$$

where

$$0 \leq x_1 \leq n_1, \quad 0 \leq x_2 \leq n_2, \quad \ldots, \quad 0 \leq x_k \leq n_k.$$

We now have upper bounds on the x_i's, and these cannot be handled in the same way as lower bounds. In Chapter 6 we show how the inclusion-exclusion principle provides a satisfactory method for this case.

3.6 Exercises

1. For each of the four combinations of the two properties (a) and (b), count the number of four-digit numbers whose digits are either $1, 2, 3, 4,$ or 5:

   (a) The digits are distinct.

   (b) The number is even.

   Note that there are four problems here: $\emptyset$ (no further restriction), $\{a\}$ (property (a) holds), $\{b\}$ (property (b) holds), $\{a, b\}$ (both properties (a) and (b) hold).

2. How many orderings are there for a deck of 52 cards if all the cards of the same suit are together?

3. In how many ways can a poker hand (5 cards) be dealt? How many different poker hands are there?

4. How many distinct positive divisors do each of the following numbers have?

   (a) $3^4 \times 5^2 \times 7^6 \times 11$

   (b) 620

   (c) 10^{10}

5. Determine the largest power of 10 that is a factor of the following numbers (equivalently, the number of terminal 0's, using ordinary base 10 representation):

   (a) 50!.

(b) 1000!.

6. How many integers greater than 5400 have both of the following properties?

   (a) The digits are distinct.

   (b) The digits 2 and 7 do not occur.

7. In how many ways can six men and six ladies be seated at a round table if the men and ladies are to sit in alternate seats?

8. In how many ways can 15 people be seated at a round table if B refuses to sit next to A? What if B only refuses to sit on A's right?

9. A committee of 4 is to be chosen from a club which boasts a membership of 10 men and 12 women. How many ways can the committee be formed if it is to contain at least 2 women? How many ways if, in addition, one particular man and one particular woman who are members of the club refuse to serve together on the committee?

10. How many sets of 3 numbers each can be formed from the numbers $\{1, 2, 3, \ldots, 20\}$ if no 2 consecutive numbers are to be in a set?

11. A football team of 11 players is to be selected from a set of 15 players, 5 of whom can only play in the backfield, 8 of whom can only play on the line, and 2 of whom can play either in the backfield or on the line. Assuming a football team has 7 men on the line and 4 in the backfield, determine the number of football teams possible.

12. There are 100 students at a school and three dormitories, A, B, and C, with capacities 25, 35 and 40 respectively.

    (a) How many ways are there to fill the dormitories?

    (b) Suppose that of the 100 students, 50 are men and 50 are women and that A is an all-men's dorm, B is an all-women's dorm, and C is co-ed. How many ways are there to fill the dormitories?

13. A classroom has 2 rows of 8 seats each. There are 14 students, 5 of whom always sit in the front row and 4 of whom always sit in the back row. In how many ways can the students be seated?

14. At a party there are 15 men and 20 women.

    (a) How many ways are there to form 15 couples consisting of one man and one woman?

    (b) How many ways are there to form 10 couples consisting of one man and one woman?

15. Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$

by using a combinatorial argument and not the values of these numbers as given in Theorem 3.3.1.

16. In how many ways can 6 indistinguishable rooks be placed on a 6-by-6 board so that no two rooks can attack another? In how many ways if there are 2 red and 4 blue rooks?

17. In how many ways can 2 red and 4 blue rooks be placed on an 8-by-8 board so that no two rooks can attack one another?

18. We are given 8 rooks, 5 of which are red and 3 of which are blue.

    (a) In how many ways can the 8 rooks be placed on an 8-by-8 chessboard so that no two rooks can attack another?

    (b) In how many ways can the 8 rooks be placed on a 12-by-12 chessboard so that no two rooks attack each other?

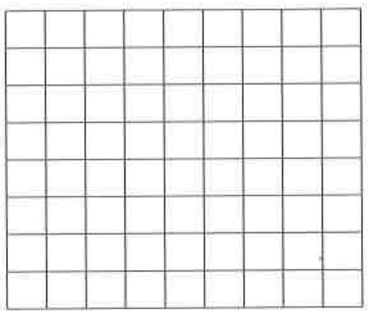
19. Determine the number of circular permutations of $\{0, 1, 2, \ldots, 9\}$ in which 0 and 9 are not opposite. (Hint: Count those in which 0 and 9 are opposite.)

20. In how many ways can 5 indistinguishable rooks be placed on an 8-by-8 chessboard so that no rook can attack another and neither the first row nor the first column is empty?

21. A secretary works in a building located 9 blocks east and 7 blocks north of his home. Every day he walks 16 blocks to work. (See the map that follows.)

    (a) How many different routes are possible for him?

    (b) How many different routes if the block in the easterly direction, which begins 4 blocks east and 3 blocks north of his home, is under water (and he can't swim)? (Hint: Count the routes that use the block under water.)



22. Let S be a multiset with repetition numbers $n_1, n_2, \ldots, n_k$ where $n_1 = 1$. Let $n = n_2 + \cdots + n_k$. Prove that the number of circular permutations of S equals

$$\frac{n!}{n_2! \cdots n_k!}.$$

23. We are to seat 5 men, 5 women, and 1 dog in a circular arrangement around a table. In how many ways can this be done if no man is to sit next to a man and no woman is to sit next to a woman?

24. In a soccer tournament of 15 teams, the top 3 teams are awarded gold, silver, and bronze cups, and the last 3 teams are dropped to a lower league. We regard two outcomes of the tournament as the same if the teams which receive the gold, silver, and bronze cups, respectively, are identical and the teams which drop to a lower league are also identical. How many different possible outcomes are there for the tournament?

25. Determine the number of 11-permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}.$$

26. Determine the number of 10-permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}.$$

27. Determine the number of 11-permutations of the multiset

$$\{3 \cdot a, 3 \cdot b, 3 \cdot c, 3 \cdot d\}.$$

28. List all 3-combinations and 4-combinations of the multiset

$$\{2 \cdot a, 1 \cdot b, 3 \cdot c\}.$$

29. Determine the total number of combinations (of any size) of a multiset of objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively.

30. A bakery sells 6 different kinds of pastry. How many different dozens of pastry can you buy (assuming you have plenty of money)? What if you buy at least one of each kind?

31. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$?.

32. There are twenty identical sticks lined up in a row occupying twenty distinct places:

$$||||||||||||||||||||,$$

and six of them are to be chosen.

    (a) How many choices are there?

    (b) How many choices are there if no two of the chosen sticks can be consecutive?

    (c) How many choices are there if there must be at least two sticks between each pair of chosen sticks?

33. There are n sticks lined up in a row and k of them are to be chosen.

    (a) How many choices are there?

(b) How many choices are there if no two of the chosen sticks can be consecutive?

(c) How many choices are there if there must be at least l sticks between each pair of chosen sticks?

34. In how many ways can 12 indistinguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one piece of fruit?

35. Determine the number of ways to distribute 10 orange drinks, 1 lemon drink, and 1 lime drink to 4 thirsty students so that each student gets at least 1 drink, and the lemon and lime drinks go to different students.

36. Determine the number of r-combinations of the multiset

$$\{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}.$$

37. Prove that the number of ways to distribute n different objects among k children equals k^n.

38. Twenty books are to be put on five book shelves, each of which holds at least twenty books.

(a) How many diffferent ways are there to arrange the twenty books on the shelves?

(b) How many different arrangements are there if you only care about the number of books on the shelves (and not which book is where)?

39. (a) There are an even number $2n$ of people at a party, and they talk together in pairs with everyone talking with someone (so n pairs). In how many different ways can the $2n$ people be talking like this?

(b) Now suppose that there are an odd number $2n+1$ of people at the party with everyone but one person talking with someone. How many different ways pairings are there?

Chapter 4

Generating Permutations and Combinations

In this chapter we explore some features of permutations and combinations that are not directly related to counting. We discuss some ordering schemes for permutations and combinations, and algorithms for carrying them out. We also introduce the idea of a relation on a set and discuss two important instances, those of partial order and equivalence relation.

4.1 Generating Permutations

The set $\{1, 2, \ldots, n\}$ consisting of the first n positive integers has $n!$ permutation which, even if n is only moderately large, is quite enormous. For instance, 15! is more than 1,000,000,000,000. A useful and readily computable approximation to $n!$ is given by *Stirling's formula*:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

where $\pi = 3.141\ldots$, and $e = 2.718\ldots$ is the base of the natural logarithm. As n grows without bound, the ratio of $n!$ to $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ approaches 1. A proof of this can be found in many texts on advanced calculus and in an article by Feller.[1]

Permutations are of importance in many different circumstances, both theoretical and applied. For sorting techniques in computer

[1] W. Feller: A direct proof of Stirling's formula, *American Mathematical Monthly*, 74 (1967), 1223-1225.