Abstract Algebra	Test #2	November 1, 2006
MATH 501		R. Hammack

Directions: This is a take-home test. It is due at the beginning of class on Wednesday, November 8.

You may discuss the problems among yourselves and share ideas, but the work you turn in must be your own. At the end of your solution to each problem, please list who (if anyone) you talked to about that problem, plus any additional information you want me to know (i.e. that you *gave* more help than you *received*, or vice versa, etc.).

- Please write all solutions completely and neatly, rewriting drafts if necessary.
- In order to get full credit, you must show all of your work and explain all of your reasoning.
- You may consult your text, notes and classmates, but **no** other source.
- Additional copies of this test can be downloaded from my web page if needed.
- All problems are weighted equally.

1. (a) Show that $H = \{2^a3^b \mid a, b \in \mathbb{Z}\}$ is a subgroup of $\mathbb{R}^+$.

    i. Note that H is closed, for if $x, y \in H$, then $x = 2^a3^b$ and $y = 2^c3^d$ for integers a, b, c and d, so $xy = 2^a3^b2^c3^d = 2^{a+c}3^{b+d} \in H$.

    ii. The identity 1 is in H because $1 = 2^03^0 \in H$.

    iii. If $x \in H$, then $x = 2^a3^b$ for integers a and b. Then $-a$ and $-b$ are also integers, so $x^{-1} = (2^a3^b)^{-1} = 2^{-a}3^{-b} \in H$.

    This shows that H is a subgroup of $\mathbb{R}^+$.

    (b) Either prove or disprove the statement $H \cong \mathbb{Z} \times \mathbb{Z}$.
    Define $\varphi : \mathbb{Z} \times \mathbb{Z} \to H$ by $\varphi(a, b) = 2^a3^b$.
    This is a homomorphism because the following shows the homomorphism property holds:

    $$\varphi((a, b) + (c, d)) = \varphi(a + c, b + d) = 2^{a+c}3^{b+d} = (2^a3^b)(2^c3^d) = \varphi(a, b)\varphi(c, d).$$

    It is onto, because if $y \in H$, then there are integers a and b for which $y = 2^a2^b = \varphi(a, b)$.
    Finally, $\ker(\varphi) = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid \varphi(a, b) = 1\} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 2^a3^b = 1\} = \{(0, 0)\}$.
    Since its kernel is trivial, φ is one-to-one (Corollary 13.18). Consequently, φ is an isomorphism, so $\mathbb{Z} \times \mathbb{Z} \cong H$.

2. List all abelian groups of order 3000.
    $3000 = 3 \cdot 2^3 \cdot 5^3$. Thus, the groups are as follows:

    (a) $\mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_{27}$
    (b) $\mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_9$
    (c) $\mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
    (d) $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{27}$
    (e) $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27}$
    (f) $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9$
    (g) $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$
    (h) $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
    (i) $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

3. In this problem G is a group of order pq where p and q are prime numbers (possibly equal).

   (a) Show that every proper subgroup of G is cyclic.
   Proof. Suppose H is a proper subgroup of G. Then, by Lagrange's Theorem, the order of H must divide the order pq of G. Hence, as p and q are prime, the order of H is either p or q. Either way, the order of H is prime. Now take an element $a \in H$ with $a \neq e$. Consider the subgroup $\langle a \rangle$ of H. The order of this subgroup must divide the order of H, which is prime. Hence $|\langle a \rangle| = 1$ or $|\langle a \rangle| = |H|$. But since a is not the identity, we know $|\langle a \rangle| \neq 1$, so it must be that $|\langle a \rangle| = |H|$. Since H is finite, it follows that $H = \langle a \rangle$, and therefore H is cyclic.

   (b) Can it be concluded that G is abelian? Explain.
   No. It could be that $G = S_3$, which has order $6 = 2 \cdot 3$ (a product of two primes), and in this case G is not abelian.

   (c) If G is abelian, can it be concluded that G is cyclic? Explain.
   No. It could be that $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which has order $4 = 2 \cdot 2$ (a product of two primes), and in this case G is not cyclic.

4. Find a subgroup of $GL(3, \mathbb{R})$ that is isomorphic to S_3. (Please write down each element of the subgroup.)

Look at what happens when you permute the three rows of I with elements of S_3:

When ρ_0 permutes the rows of I, you get $P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

When ρ_1 permutes the rows of I, you get $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

When ρ_2 permutes the rows of I, you get $P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

When μ_1 permutes the rows of I, you get $M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

When μ_2 permutes the rows of I, you get $M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

When μ_3 permutes the rows of I, you get $M_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Let $H = \{P_0, P_1, P_2, M_1, M_2, M_3\}$

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Doing the matrix multiplications, you find the table for H is as follows.

	P_0	P_1	P_2	M_1	M_2	M_3
P_0	P_0	P_1	P_2	M_1	M_2	M_3
P_1	P_1	P_2	P_2	M_3	M_1	M_2
P_2	P_2	P_0	P_2	M_2	M_3	M_1
M_1	M_1	M_2	M_3	P_0	P_1	P_2
M_2	M_2	M_3	M_1	P_2	P_0	P_1
M_3	M_3	M_1	M_2	P_1	P_2	P_0

Since this has the same structure as the table for S_3, we have $H \cong S_3$.

5. Suppose $\varphi : G \to K$ is a homomorphism with kernel H, and a is a fixed element of G. Let $X = \{x \in G \mid \varphi(x) = \varphi(a)\}$. Show $X = Ha$.

Proof. First suppose $x \in X$. This means $\varphi(x) = \varphi(a)$. From this we get

$$\begin{aligned}
\varphi(x) &= \varphi(a) \\
\varphi(x)\varphi(a)^{-1} &= \varphi(a)\varphi(a)^{-1} \\
\varphi(x)\varphi(a)^{-1} &= e_K \\
\varphi(x)\varphi(a^{-1}) &= e_K \\
\varphi(xa^{-1}) &= e_K
\end{aligned}$$

whence it follows that $xa^{-1} \in \ker(\varphi) = H$. Consequently $xa^{-1} = h$ for some $h \in H$, and therefore $x = ha$, which means $x \in Ha$. This shows $X \subseteq Ha$.

Next, suppose $x \in Ha$, so $x = ha$ for some $h \in H$.

$$\begin{aligned}
x &= ha & & \\
\varphi(x) &= \varphi(ha) & & (\text{ take } \varphi \text{ of both sides}) \\
\varphi(x) &= \varphi(h)\varphi(a) & & (\varphi \text{ is a homo.}) \\
\varphi(x) &= e_k\varphi(a) & & (h \in H = \ker(\varphi)) \\
\varphi(x) &= \varphi(a) & &
\end{aligned}$$

But $\varphi(x) = \varphi(a)$ means $x \in X$, so $Ha \subseteq X$.

Since $X \subseteq Ha$ and $Ha \subseteq X$, we have $X = Ha$.

6. (a) List the cosets of the subgroup $\langle(1,2)\rangle$ of $\mathbb{Z}_4 \times \mathbb{Z}_8$.

Note $H = \langle(1,2)\rangle = \{(0,0),(1,2),(2,4),(3,6)\}$, so the cosets are:

$(0,0) + H = \{(0,0),(1,2),(2,4),(3,6)\}$
$(0,1) + H = \{(0,1),(1,3),(2,5),(3,7)\}$
$(0,2) + H = \{(0,2),(1,4),(2,6),(3,0)\}$
$(0,3) + H = \{(0,3),(1,5),(2,7),(3,1)\}$
$(0,4) + H = \{(0,4),(1,6),(2,0),(3,2)\}$
$(0,5) + H = \{(0,5),(1,7),(2,1),(3,3)\}$
$(0,6) + H = \{(0,6),(1,0),(2,2),(3,4)\}$
$(0,7) + H = \{(0,7),(1,1),(2,3),(3,5)\}$

(b) What familiar group is $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$ isomorphic to?
Notice that from the above list, we see that
$\langle(0,1) + H\rangle = \{(0,0)+H,(0,1)+H,(0,2)+H,(0,3)+H,(0,4)+H,(0,5)+H,(0,6)+H,(0,7)+H\}$,
So $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$ is a cyclic group with 8 elements, so it is isomorphic to $\mathbb{Z}_8$.

7. Note that $\{1,-1\}$ is a subgroup of $\mathbb{R}^*$. Moreover, it is a normal subgroup because $\mathbb{R}^*$ is abelian. What familiar group is $\mathbb{R}^*/\{1,-1\}$ isomorphic to? Consider using the Fundamental Homomorphism Theorem.

Consider the map $\varphi : \mathbb{R}^* \to \mathbb{R}^+$ defined as $\varphi(x) = |x|$. This is a homomorphism because $\varphi(xy) = |xy| = |x||y| = \varphi(x)\varphi(y)$. Moreover, it is an onto homomorphism because given any $y \in \mathbb{R}^+$, it is also the case that $y \in \mathbb{R}^*$, and $\varphi(y) = y$. Also, it is clear that the kernel of φ is $\{1,-1\}$. By the Fundamental Homomorphism Theorem, there is an isomorphism $\mu : \mathbb{R}^*/\{1,-1\} \to \mathbb{R}^+$, so $\mathbb{R}^*/\{1,-1\} \cong \mathbb{R}^+$.

8. (a) Give an explicit example of a subgroup that is *not* normal.

   Consider the subgroup $H = \{\rho_0, \mu_1\}$ of S_3.
   Notice $\mu_2 H = \{\mu_2, \rho_2\}$,
   and $H\mu_2 = \{\mu_2, \rho_1\}$.
   Since $\mu_2 H \neq H\mu_2$, it follows that H is not normal.

   (b) Show that A_n is a normal subgroup of S_n.
   Recall that A_n consists of all the even permutations of S_n.
   Theorem 14.13 says that A_n will be normal provided $\sigma\tau\sigma^{-1} \in A_n$ for all $\sigma \in S_n$ and $\tau \in A_n$.
   Now τ is even. And σ and σ^{-1} are either both even or both odd. Hence $\sigma\tau\sigma^{-1}$ is even, so $\sigma\tau\sigma^{-1} \in A_n$.
   Thus A_n is normal.

   (c) What familiar group is S_n/A_n isomorphic to?

   Since S_n/A_n has exactly two elements, it follows that $S_n/A_n \cong \mathbb{Z}_2$.

9. Suppose H is a normal subgroup of a group G, and H has the additional property that $aba^{-1}b^{-1} \in H$ for any $a, b \in G$. Show G/H is abelian.

   Given cosets aH and bH of G/H, we wish to show $(aH)(bH) = (bH)(aH)$. Now, since $aba^{-1}b^{-1} \in H$ we have $aba^{-1}b^{-1}H = eH$. Then using the definition of the binary operation in G/H, we have

$$
\begin{aligned}
aba^{-1}b^{-1}H &= eH \\
(abH)(a^{-1}b^{-1}H) &= eH \\
(abH)((ba)^{-1}H) &= eH \\
(abH)(baH)^{-1} &= eH \\
(abH)(baH)^{-1}(baH) &= (eH)(baH) \\
(abH) &= (baH) \\
(aH)(bH) &= (bH)(aH)
\end{aligned}
$$

   Therefore G/H is abelian.

10. Show that if H and K are normal subgroups of a group G and $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$. (Suggestion: Is the equation $h(kh^{-1}k^{-1}) = (hkh^{-1})k^{-1}$ true? What does it imply?)

    Proof. Suppose $h \in H$ and $k \in K$.
    Recall that a subgroup H is normal if and only if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$ (Theorem 13.14). Thus:
    Since H is normal, we have $kh^{-1}k^{-1} \in H$, and since H is closed, $h(kh^{-1}k^{-1}) \in H$.
    Since K is normal, we have $hkh^{-1} \in K$, and since K is closed, $(hkh^{-1})k^{-1} \in K$.
    By the associative property, we have $hkh^{-1}k^{-1} = h(kh^{-1}k^{-1}) = (hkh^{-1})k^{-1}$,
    so the previous lines imply $hkh^{-1}k^{-1}$ is in both H and K.
    But, as $H \cap K = \{e\}$, it must follow that $hkh^{-1}k^{-1} = e$, whence

$$
\begin{aligned}
hkh^{-1}k^{-1} &= e \\
hkh^{-1}k^{-1}k &= ek \\
hkh^{-1} &= k \\
hkh^{-1}h &= kh \\
hk &= kh
\end{aligned}
$$