Abstract Algebra | Test #2 | November 1, 2006

MATH 501 | R. Hammack

Directions: This is a take-home test. It is due at the beginning of class on Wednesday, November 8.

You may discuss the problems among yourselves and share ideas, but the work you turn in must be your own. At the end of your solution to each problem, please list who (if anyone) you talked to about that problem, plus any additional information you want me to know (i.e. that you *gave* more help than you *received*, or vice versa, etc.).

- Please write all solutions completely and neatly, rewriting drafts if necessary.
- In order to get full credit, you must show all of your work and explain all of your reasoning.
- You may consult your text, notes and classmates, but **no** other source.
- Additional copies of this test can be downloaded from my web page if needed.
- All problems are weighted equally.

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1. (a) Show that $H = \{2^a 3^b \mid a, b \in \mathbb{Z}\}$ is a subgroup of $\mathbb{R}^+$.
   (b) Either prove or disprove the statement $H \cong \mathbb{Z} \times \mathbb{Z}$.

2. List all abelian groups of order 3000.

3. In this problem G is a group of order pq where p and q are prime numbers (possibly equal).
   (a) Show that every proper subgroup of G is cyclic.
   (b) Can it be concluded that G is abelian? Explain.
   (c) If G is abelian, can it be concluded that G is cyclic? Explain.

4. Find a subgroup of $GL(3, \mathbb{R})$ that is isomorphic to S_3. (Please write down each element of the subgroup.)

5. Suppose $\varphi : G \to K$ is a homomorphism with kernel H, and a is a fixed element of G.
   Let $X = \{x \in G \mid \varphi(x) = \varphi(a)\}$. Show $X = Ha$.

6. (a) List the cosets of the subgroup $\langle(1, 2)\rangle$ of $\mathbb{Z}_4 \times \mathbb{Z}_8$.
   (b) What familiar group is $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle$ isomorphic to?

7. Note that $\{1, -1\}$ is a subgroup of $\mathbb{R}^*$. Moreover, it is a normal subgroup because $\mathbb{R}^*$ is abelian.
   What familiar group is $\mathbb{R}^*/\{1, -1\}$ isomorphic to? Consider using the Fundamental Homomorphism Theorem.

8. (a) Give an explicit example of a subgroup that is *not* normal.
   (b) Show that A_n is a normal subgroup of S_n.
   (c) What familiar group is S_n/A_n isomorphic to?

9. Suppose H is a normal subgroup of a group G, and H has the additional property that $aba^{-1}b^{-1} \in H$ for any $a, b \in G$. Show G/H is abelian.

10. Show that if H and K are normal subgroups of a group G and $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$. (Suggestion: Is the equation $h(kh^{-1}k^{-1}) = (hkh^{-1})k^{-1}$ true? What does it imply?)