Abstract Algebra	Test #1	October 7, 2010
MATH 501	R. Hammack	Name ____________

Directions: Answer each question in the space provided. Use of any electronic device (calculators, i-pods, etc.) is not allowed during this test.

1. (30 points) Short Answer. You do not need to show your work for problems on this page.

(a) List the generators of $\mathbb{Z}_{12}$ .... $\boxed{1, 5, 7, 11}$

(b) Write a multiplication table for $U(12)$. ....

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

(c) $|A_n| =$ .... $\boxed{\frac{n!}{2}}$

(d) A group has 45 elements. What are the possible orders of its subgroups? .... $\boxed{1, 3, 5, 9, 15, 45}$

(e) Suppose a is a generator of a cyclic group G. Give a generator for the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$ .... $\boxed{a^{\text{lcm}(m,n)}}$

(f) Give an example of a nontrivial abelian subgroup of a non-abelian group. .... $\boxed{A_3 \subseteq S_3}$

(g) Write $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 6 & 7 & 2 & 4 & 3 \end{pmatrix} \in S_7$ as a product of disjoint cycles. .... $\boxed{(152)(3647)}$

(h) Is the permutation μ from part (h) even or odd? .... $\boxed{\text{Odd}}$ because (152) is even and (3647) is odd.

(i) Find the order of the permutation μ from part (h). .... $\boxed{\text{lcm}(3,4)=12}$

(j) Write the following as a product of disjoint cycles: $(215)(3142)^{-1}$ .... $\boxed{(13)(245)}$

2. (10 points) List the left cosets of the subgroup $H = \{(1), (13)\}$ of S_3.

$$\begin{aligned} (1)H &= \{(1), (13)\} \\ (12)H &= \{(12), (321)\} \\ (23)H &= \{(23), (123)\} \end{aligned}$$

3. (10 points) Find all possible orders of elements in S_7.

In the table below, examples of permutations (as cycles or products of disjoint cycles) in S_7 are paired with their orders.

Permutation	order
(1)	1
(1 2)	2
(1 2 3)	3
(1 2 3 4)	4
(1 2 3 4 5)	5
(1 2 3 4 5 6)	6
(1 2 3 4 5 6 7)	7
(1 2)(3 4 5 6 7)	lcm(2,5)=10
(1 2 3)(4 5 6 7)	lcm(3,4)=12

In trying other combinations of disjoint cycles, we quickly see that the above table captures all possible orders. For example, (12)(3458) has order gcd(2,4)=4, and this order already appears on the table, etc.

Thus the possible orders are $1, 2, 3, 4, 5, 6, 7, 10$ and 12.

4. (10 points) Prove that if a group G has no proper nontrivial subgroups, then G is cyclic.

Proof. Suppose G has no proper nontrivial subgroups. Take an element $a \in G$ for which $a \neq e$. Consider the cyclic subgroup $\langle a \rangle$. This subgroup contains at least e and a, so it is not trivial. But G has no proper subgroups, so it must be that $\langle a \rangle = G$. Thus G is cyclic, by definition of a cyclic group. ■

5. (10 points) Suppose a group G has the property that $a^2 = e$ for every $a \in G$. Prove that G is abelian.

Proof. Suppose G has the property that $a^2 = e$ for every $a \in G$. Take arbitrary elements $x, y \in G$. By assumption we have $(xy)^2 = e$. We work with this as follows.

$$\begin{aligned}
(xy)^2 &= e \\
xyxy &= e \\
x(xyxy) &= xe \\
(xx)(yxy) &= x \\
e(yxy) &= x \\
yxy &= x \\
(yxy)y &= xy \\
(yx)(yy) &= xy \\
(yx)e &= xy \\
yx &= xy
\end{aligned}$$

This establishes $xy = yx$, so G is abelian. ■

6. (10 points) Let g be an element of a group G, and define a map $\lambda_g : G \to G$ as $\lambda_g(x) = gx$.
Show that λ_g is a permutation of G.

Proof. By definition, a permutation of G is just a bijection $G \to G$. Thus we only need to show that λ_g is bijective.

First we show λ_g is injective. Suppose $\lambda_g(x) = \lambda_g(y)$. This means $gx = gy$. Multiplying both sides by g^{-1} gives $x = y$. It follows that λ_g is injective.

Next let's show that λ_g is surjective. Take an arbitrary $a \in G$. Now, since $g \in G$, we must also have $g^{-1} \in G$, hence $g^{-1}a \in G$. Now observe that $\lambda_g(g^{-1}a) = gg^{-1}a = a$. Thus λ_g is surjective.

Since it is both injective and surjective, λ_g is bijective, and therefore it's a permutation of G. ■

7. (10 points) Let G be an **abelian** group. Show that the set of elements of finite order in G form a subgroup.

Proof. Let $H = \{a : a \in G \text{ and } a \text{ has finite order } \} \subseteq G$. We need to show that H is a subgroup of G.

1. Since $e^1 = e$, the identity e has (finite) order 1, so $e \in H$.
2. Suppose $a, b \in H$. This means a and b have finite orders, say m and n, respectively. Therefore $a^m = e$ and $b^n = e$. Now consider the product ab. Using the usual laws of exponents and the fact that G is abelian, we get

$$(ab)^{mn} = \underbrace{(ab)(ab)\cdots(ab)}_{mn \text{ times}} = \underbrace{aaa\cdots a}_{mn}\underbrace{bbb\cdots b}_{mn} = a^{mn}b^{mn} = (a^m)^n(b^n)^m = e^n e^m = e.$$

Therefore $(ab)^{mn} = e$, so ab has finite order, and is therefore in H. This proves that H is closed under multiplication.

3. Suppose $a \in H$, so a has finite order; say $a^m = e$. Then

$$(a^{-1})^m = a^{-m} = (a^m)^{-1} = e^{-1} = e,$$

which means a^{-1} has finite order. Thus $a^{-1} \in H$, so H is closed with respect to taking inverses.

The above considerations show that H satisfies the conditions of Theorem 3.9, so H is a subgroup of G. ■

8. (10 points) Suppose H is a subgroup of a group G, and $[G : H] = 2$.
Suppose also that a and b are in G, but not in H. Show that $ab \in H$.

Proof. Suppose $[G : H] = 2$ and $a, b \notin H$. Now, since $a \notin H$, it follows that $a^{-1} \notin H$. (Otherwise, if a^{-1} were in H, its inverse a would be in H, and this is not the case.)

Since neither a^{-1} nor b is in H, we know that $a^{-1}H \neq H$ and $bH \neq H$.

But since $[G : H] = 2$, we know that H has only two left cosets in G. One of these cosets is H. By the previous paragraph, $a^{-1}H$ and bH must both be equal to the coset that is not H, and therefore $a^{-1}H = bH$.

Now, consider an arbitrary element of $a^{-1}H$, which has form $a^{-1}h$ for some $h \in H$. Since this is also an element of bH, it must also equal bh' for some $h' \in H$. Therefore we have

$$a^{-1}h = bh'.$$

Multiply both sides of this by a (on the left) to get $h = abh'$. Now multiply both sides by the inverse of h' (on the right) to get

$$ab = h(h')^{-1} \in H.$$

This completes the proof. ■