Abstract Algebra — Test #1 — October 2, 2006

Name: ____________________ R. Hammack Score: ______

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is **not** allowed on this test.

1. (12 points) Here is a partial table for a **commutative** and **associative** binary operation on a set $S = \{a, b, c, d\}$.

$*$	a	b	c	d
a	d	c		
b		d		
c		a		
d				

Supply the following information.

(a) $b * a = a * b = c$

(b) $b * c = c * b = a$

(c) $d * a = (b * b) * a = b * (b * a) = b * c = a$

(d) $d * c = (b * b) * c = b * (b * c) = b * a = c$

2. (18 points) For each of the following binary structures, say which are groups and which are not. If a structure is not a group, state which (if any) of the group axioms $\mathscr{G}_1$, $\mathscr{G}_2$, or $\mathscr{G}_3$ fail.

(a) $\langle \mathbb{Z}_5, +_5 \rangle$ **Group**

(b) $\langle \mathbb{Q}^*, \div \rangle$ **Not a group**; $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$ all fail.

(c) $\langle \mathbb{Q}, + \rangle$ **Group**

(d) $\langle \mathbb{Z}^*, \cdot \rangle$ **Not a group**; $\mathscr{G}_3$ fails.

(e) $\langle U, \cdot \rangle$ **Group**

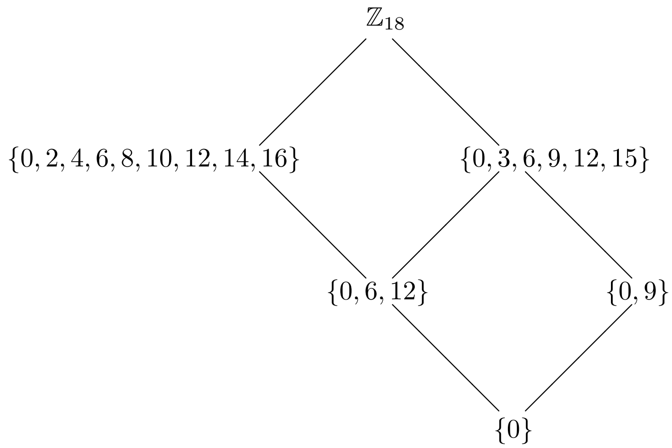
(f) $\langle 3\mathbb{Z}, + \rangle$ **Group**

(g) The set of all one-to-one and onto functions $f : \mathbb{R} \to \mathbb{R}$ under the binary operation of function composition. **Group**

(h) $M_{3\times 3}(\mathbb{R})$ under matrix multiplication. **Not a group**; $\mathscr{G}_3$ fails.

(i) $M_{3\times 3}(\mathbb{R})$ under matrix addition. **Group**

3. (10 points) Draw the subgroup lattice for $\mathbb{Z}_{18}$.



4. (5 points) List the generators of $\mathbb{Z}_{18}$.

Generators: 1, 5, 7, 11, 13, 17

5. (5 points) List the elements of the subgroup $\langle 12, 9 \rangle$ of $\mathbb{Z}_{18}$.

$\{0, 3, 6, 9, 12, 15\}$

6. (5 points) Suppose $\varphi : \mathbb{Z}_8 \to U_8$ is an isomorphism satisfying $\varphi(3) = \dfrac{1+i}{\sqrt{2}}$. Find $\varphi(6)$.

$$\varphi(6) = \varphi(3+3) = \varphi(3) \cdot \varphi(3) = \frac{1+i}{\sqrt{2}} \cdot \frac{1+i}{\sqrt{2}} = \frac{1+i+i-1}{2} = \boxed{i}$$

7. (5 points) Suppose H is a proper subgroup of G. (Proper means $H \subseteq G$ but $H \neq G$.)
Is it possible that $H \cong G$?
If this is possible, give an example of such a G and H. If it's not possible, say why.

Yes this is quite possible. Consider $H = 2\mathbb{Z}$ and $G = \mathbb{Z}$ and note that H is a proper subgroup of G.
Also, $H \cong G$, for the function $\varphi : G \to H$ defined as $\varphi(n) = 2n$ is an isomorphism:
It's one-to-one, for if $\varphi(m) = \varphi(n)$, then $2m = 2n$, so $m = n$.
It's onto, for if $y \in H$, then $y = 2k$ for some $k \in \mathbb{Z}$, and $\varphi(k) = 2k = y$.
Finally, note that $\varphi(m+n) = 2(m+n) = 2m + 2n = \varphi(m) + \varphi(n)$, so the homomorphism property holds.

8. (10 points) Suppose K is a subgroup of an abelian group G.
Show that the set $H = \{x \in G \mid x^2 \in K\}$ is a subgroup of G.

Proof. We make the following observations:

(a) First we show H is closed. Suppose $a, b \in H$, which means $a^2 = e$ and $b^2 = e$. Using this with the fact that G is abelian we get $(ab)^2 = (ab)(ab) = abab = aabb = a^2b^2 = ee = e$. Now, the fact that $(ab)^2 = e$ means $ab \in H$, so H is closed.

(b) Observe $e \in H$ because $e^2 = e$ means e satisfies the requirement for being in H.

(c) Suppose $a \in H$. This means $a^2 = e$, or $aa = e$. Thus $a^{-1} = a \in H$.

Considerations (a), (b) and (c) above show that H is a subgroup.

9. (10 points) Suppose a nonempty finite subset H of a group G is closed under the binary operation of G.
Prove that H is a subgroup of G.

Proof. We are given that H is closed, so we do not need to prove that particular condition for H being a subgroup.

Next we show $e \in H$. Suppose $|H| = m$. Since H is nonempty, there is some element $a \in H$. Consider the list $a^1, a^2, a^3 \dots a^{m+1}$. Since H is closed, each element on this list is in H. Also since the length of the list is one more than $|H|$, the list has at least one repeated item. Thus $a^r = a^s$ for some integers r and s with $1 \leq r < s \leq m+1$. Multiplying both sides of this equation by a^{-r} gives $e = a^{s-r}$. Consequently, the $(s-r)th$ element of the list is e, which means $e \in H$.

Suppose a is an arbitrary element of H. The above paragraph shows that there is a positive integer k with $a^k = e$. Then $aa^{k-1} = e$, so $a^{-1} = a^{k-1}$. But $a^{k-1} \in H$ because $a \in H$ and H is closed. Therefore $a^{-1} \in H$.

The previous three paragraphs show that H is a subgroup of G.

10. (10 points) Prove that if a group G is cyclic, then G is abelian.

Proof. Suppose G is cyclic, so $G = \langle a \rangle = \{a^n | n \in \mathbb{Z}\}$ for some $a \in G$.
Therefore, given two arbitrary elements x and y of G, there are integers m and n with $x = a^m$ and $y = a^n$.
Consequently $xy = a^m a^n = a^{m+n} = a^n a^m = yx$, which means G is abelian.

11. (10 points) Suppose g is one fixed element of a group G. Define the function $\varphi : G \to G$ as $\varphi(x) = gxg^{-1}$. Prove that φ is an isomorphism from G to itself.

Proof. First notice that φ is one-to-one: If $\varphi(x) = \varphi(y)$, then $gxg^{-1} = gyg^{-1}$.
Left-multiplying both sides by g^{-1} gives $xg^{-1} = yg^{-1}$.
Now right-multiplying both sides by g produces $x = y$. This shows φ is one-to-one.

Next observe that φ is onto, for if $y \in G$, let $x = g^{-1}yg$ (which is also in G).
Observe $\varphi(x) = \varphi(g^{-1}yg) = gg^{-1}ygg^{-1} = y$, so φ is onto.

Finally, note that the homomorphism property holds:
$\varphi(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = (gxg^{-1})(gyg^{-1}) = \varphi(x)\varphi(y)$.

It follows that φ is an isomorphism.