Name: ______________________ R. Hammack Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is **not** allowed on this test.

1. (12 points) Here is a partial table for a **commutative** and **associative** binary operation on a set $S = \{a, b, c, d\}$.

$*$	a	b	c	d
a	d	c		
b		d		
c		a		
d				

Supply the following information.

(a) $b * a =$

(b) $b * c =$

(c) $d * a =$

(d) $d * c =$

2. (18 points) For each of the following binary structures, say which are groups and which are not. If a structure is not a group, state which (if any) of the group axioms $\mathscr{G}_1$, $\mathscr{G}_2$, or $\mathscr{G}_3$ fail.

(a) $\langle \mathbb{Z}_5, +_5 \rangle$

(b) $\langle \mathbb{Q}^*, \div \rangle$

(c) $\langle \mathbb{Q}, + \rangle$

(d) $\langle \mathbb{Z}^*, \cdot \rangle$

(e) $\langle U, \cdot \rangle$

(f) $\langle 3\mathbb{Z}, + \rangle$

(g) The set of all one-to-one and onto functions $f : \mathbb{R} \to \mathbb{R}$ under the binary operation of function composition.

(h) $M_{3\times 3}(\mathbb{R})$ under matrix multiplication.

(i) $M_{3\times 3}(\mathbb{R})$ under matrix addition.

3. (10 points) Draw the subgroup lattice for $\mathbb{Z}_{18}$.

4. (5 points) List the generators of $\mathbb{Z}_{18}$.

5. (5 points) List the elements of the subgroup $\langle 12, 9\rangle$ of $\mathbb{Z}_{18}$.

6. (5 points) Suppose $\varphi : \mathbb{Z}_8 \to U_8$ is an isomorphism satisfying $\varphi(3) = \dfrac{1+i}{\sqrt{2}}$. Find $\varphi(6)$.

7. (5 points) Suppose H is a proper subgroup of G. (Proper means $H \subseteq G$ but $H \neq G$.)
   Is it possible that $H \cong G$?
   If this is possible, give an example of such a G and H. If it's not possible, say why.

8. (10 points) Suppose K is a subgroup of an abelian group G.
Show that the set $H = \{x \in G \mid x^2 \in K\}$ is a subgroup of G.

9. (10 points) Suppose a nonempty finite subset H of a group G is closed under the binary operation of G.
Prove that H is a subgroup of G.

10. (10 points) Prove that if a group G is cyclic, then G is abelian.

11. (10 points) Suppose g is one fixed element of a group G. Define the function $\varphi : G \to G$ as $\varphi(x) = gxg^{-1}$. Prove that φ is an isomorphism from G to itself.