MATH 501: Abstract Algebra | Test#2 | November 18, 2010

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Directions: This is a closed-book, closed-notes test. Please answer in the space provided. Explain your reasoning. Use calculators, computers, etc, is not permitted on this test.

1. Find the order of $(8, 8, 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$.

   The order of 8 in $\mathbb{Z}_{10}$ is 5.
   The order of 8 in $\mathbb{Z}_{24}$ is 3.
   The order of 8 in $\mathbb{Z}_{80}$ is 10.

   Thus the order of $(8, 8, 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$ is $\operatorname{lcm}(5, 3, 10) = \mathbf{30}$.

2. Prove or disprove: $\mathbb{Z} \cong \mathbb{Q}$.

   This is false. To see why, note that $\mathbb{Z} = \langle 1 \rangle$ is cyclic.

   But $\mathbb{Q}$ is not cyclic, because if a is any element of Q, then $\langle a \rangle = \{ka : k \in \mathbb{Z}\}$.
   This set is all integer multiples of a. It cannot possibly equal all of $\mathbb{Q}$ because, for instance, $\frac{a}{2} \notin \langle a \rangle$.

   Since $\mathbb{Z}$ is cyclic but $\mathbb{Q}$ is not, we have $\mathbb{Z} \ncong \mathbb{Q}$.

3. Suppose H is a normal subgroup of a group G. Prove or disprove:
   If H and G/H are both abelian, then G is also abelian.

   This is false, as the following counterexample shows:
   Consider $G = S_3$ and $H = A_3$. Then G is non-ablelian.
   But the subgroup $H = A_3$ of G is abelian, as $A_3 \cong \mathbb{Z}_3$.

   Moreover, A_3 is normal in S_3 as follows: The set A_3 is by definition the set of even permutations in S_3. It follows that for any permutation $\sigma \in S_3$, any permutation in $\sigma A_3 \sigma^{-1}$ is also even, for such a permutation has the form $\sigma\alpha\sigma^{-1}$. Since σ and σ^{-1} are either both even or both odd, and α is even, it follows that $\sigma\alpha\sigma^{-1}$ is even. Thus $\sigma A_3 \sigma^{-1}$ is a set of even permutations in S_3, that is so $\sigma A_3 \sigma^{-1} \subseteq A_3$.This implies that A_3 is normal.

   Now, since $|S_3| = 6$ and $|A_3| = 3$, we have $|G/H| = |S_3/A_3| = 6/3 = 2$.
   Since G/H has just two elements, it is isomorphic to $\mathbb{Z}_2$, and it thus abelian.

4. List all abelian groups of order 360, up to isomorphism.

   Note $360 = 2^3 \cdot 3^2 \cdot 5$. By the fundamental theorem of abelian groups, the possibilities are:

   $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
   $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
   $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
   $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
   $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
   $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

5. Suppose R and S are rings with multiplicative identities $1_R \in R$ and $1_S \in S$.
Prove that if $\varphi : R \to S$ is a surjective ring homomorphism, then $\varphi(1_R) = 1_S$.

Proof We must show for any $b \in S$ that $\varphi(1_R)b = b$ and $b\varphi(1_R) = b$.
Since φ is surjective, we have $b = \varphi(a)$ for some $a \in R$.
Then $\varphi(1_R)b = \varphi(1_R)\varphi(a) = \varphi(1_R a) = \varphi(a) = b$.
Likewise $b\varphi(1_R) = \varphi(a)\varphi(1_R) = \varphi(a1_R) = \varphi(a) = b$. ■

6. Suppose G and H are groups. Prove that $G \times H \cong H \times G$.

Consider the map $\varphi : G \times H \to H \times G$, defined as $\varphi((x, y)) = (y, x)$.

Note that this in injective, as follows: Suppose $\varphi((x, y)) = \varphi((a, b))$. Then $(y, x) = (b, a)$, so $y = b$ and $x = a$.
Therefore $(x, y) = (a, b)$, and it follows that φ is injective.

To see that φ is surjective, just take an arbitrary element $(x, y) \in H \times G$, and note $\varphi((y, x)) = (x, y)$.
Therefore φ is surjective.

The above two paragraphs imply that φ is bijective.

To finish the proof, observe that $\varphi\big((x, y)(a, b)\big) = \varphi((xa, yb)) = (yb, xa) = (y, x)(b, a) = \varphi((x, y))\varphi((a, b))$. ■

7. Describe all the homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_6$.

First observe that given any fixed $a \in \mathbb{Z}_6$ we can define a function $\varphi : \mathbb{Z} \to \mathbb{Z}_6$ as $\varphi(n) = n \cdot a$.
(Where $n \cdot a$ means a plus itself n times.)
This is a homomorphism because $\varphi(m+n) = (m+n) \cdot a = m \cdot a + n \cdot a = \varphi(m) + \varphi(n)$.
Let's call this homomorphism φ_a, that is, φ_a is the homomprphism for which $\varphi_a(n) = n \cdot a$.

The six different choices for a give us the following six homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_6$:

function	rule
φ_0	$\varphi_0(n) = n \cdot 0$
φ_1	$\varphi_1(n) = n \cdot 1$
φ_2	$\varphi_2(n) = n \cdot 2$
φ_3	$\varphi_3(n) = n \cdot 3$
φ_4	$\varphi_4(n) = n \cdot 4$
φ_5	$\varphi_5(n) = n \cdot 5$

No two of these functions are the same, since they all give different values when you plug in 1.
Thus so far we have six homomorphisms.

To show that these are the only six homomorphisms, we need to check that any given homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_6$ is one of the ones listed above. Given such a homomorphism, let $\varphi(1) = a \in \mathbb{Z}_6$. Then

$$\varphi(n) = \varphi(\underbrace{1+1+\ldots+1}_{n \text{ times}}) = \underbrace{\varphi(1)+\varphi(1)+\cdots+\varphi(1)}_{n \text{ times}} = n \cdot \varphi(1) = n \cdot a.$$

Therefore we have $\varphi(n) = n \cdot a$, so φ is one of the six listed homomorphisms.

$\boxed{\text{Conclusion: There are only six homomorphisms from } \mathbb{Z} \text{ to } \mathbb{Z}_6\text{. They are the ones listed above.}}$

8. Prove that if $\varphi : G \to H$ is a group homomorphism and G is cyclic, then the subgroup $\varphi(G)$ is cyclic.

Proof Suppose G is cyclic, so $G = \langle a \rangle = \{a^k : k \in \mathbb{Z}\}$ for some $a \in G$.

To show $\varphi(G)$ is cyclic, we are going to show $\varphi(G) = \langle \varphi(a) \rangle$.

Certainly, since $\varphi(a) \in \varphi(G)$, we have $\langle \varphi(a) \rangle \subseteq \varphi(G)$.

To establish the reverse inclusion, suppose $b \in \varphi(G)$. Take $c \in G$ for which $\varphi(c) = b$. Then, as a generates G, we have $c = a^k$ for some integer k. Thus $b = \varphi(c) = \varphi(a^k) = \varphi(a)^k \in \langle \varphi(a) \rangle$. This shows that $b \in \varphi(G)$ implies $b \in \langle \varphi(a) \rangle$, so $\varphi(G) \subseteq \langle \varphi(a) \rangle$.

The previous two paragraphs show $\varphi(G) = \langle \varphi(a) \rangle$, which means $\varphi(G)$ is cyclic. ■

9. If a and b are elements in a ring R, then $a(-b) = -(ab)$.

First note that $a0 = 0$ for any $a \in R$. To see why this is true, note that $a0 = a(0+0) = (a0 + a0)$. Thus we get the equation $a0 = a0 + a0$. Add $-(a0)$ to both sides of this, and we get $-(a0) + a0 = -a0 + a0 + a0$, which reduces to $0 = a0$, as desired.

Using this, now reason as follows:

$$\begin{aligned} a(-b) &= a(-b) + 0 \\ &= a(-b) + ab - (ab) \\ &= a(-b+b) - (ab) \\ &= a0 - (ab) \\ &= 0 + -(ab) \\ &= -(ab) \end{aligned}$$

■

10. Suppose R is an integral domain whose only ideals are $\{0\}$ and R. Prove that R must be a field.

An integral domain is a commutative ring with 1 and no zero divisors. To show that it is a field, we just need to show that for any nonzero element $a \in R$, there is an element $a^{-1} \in R$ for which $aa^{-1} = 1$.

Thus suppose $a \in R$, and $a \neq 0$. Consider the principal ideal $\langle a \rangle = \{ar : r \in R\}$. This ideal contains $a1 = a \neq 0$, so it is not the ideal $\{0\}$. The only other possibility is $\langle a \rangle = \{ar : r \in R\} = R$. This means that $1 \in \{ar : r \in R\}$, and consequently there is an element $r \in R$ for which $ar = 1$. Therefore a has an inverse.

This completes the proof. ■