Linear Algebra Test #3 (Chapter 3) October 9, 2006

Name: ______________________ R. Hammack Score: ________

Directions: This is a take-home test. It is due at the beginning of class on Wednesday, October 11. Please answer all questions in the space provided. Consider working the problems on scratch paper, then rewriting them neatly on the test. Additional copies of this test can be downloaded from my web page if needed.

For this test, you may discuss the problems among yourselves and share ideas, but the work you turn in must be your own (not copied). At the end of your solution to each problem, please list who (if anyone) you talked to about that problem, plus any additional information you want me to know (i.e. that you *gave* more help than you *received*, or vice versa, etc.).

- In order to get full credit, you must show all of your work.
- When appropriate, indicate your final answer clearly by putting it in a $\boxed{\text{box}}$.
- Constants that are not integers should be expressed as fractions.
- You may consult your text and notes, but **no** other source.
- Each problem is worth 10 points.

1. Use any method discussed in class to find $\begin{vmatrix} 2 & -1 & -2 & 5 \\ 3 & 3 & 6 & 1 \\ 0 & 1 & 3 & 0 \\ -2 & 4 & 8 & -2 \end{vmatrix}$.

   First do $C_3 - 2C_2 \to C_3$ and then expand along the third column:

$$\begin{vmatrix} 2 & -1 & 0 & 5 \\ 3 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -2 & 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 5 \\ 3 & 3 & 1 \\ -2 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 5 \\ 3 & 3 & 1 \\ 0 & 3 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 3 & 3 \end{vmatrix} - 3\begin{vmatrix} -1 & 5 \\ 3 & 3 \end{vmatrix} = 2 \cdot 6 + 3 \cdot 18 = \boxed{66}.$$

2. Use any method discussed in class to find the determinant of $A = \begin{bmatrix} 1 & 0 & 1 & 4 & -4 \\ 2 & 0 & 3 & 9 & 1 \\ 7 & 3 & 5 & 9 & 0 \\ -1 & 0 & 2 & 8 & 2 \\ -1 & 0 & 0 & 2 & 1 \end{bmatrix}$.

   Expand along the second column, and do the usual row reductions and the determinant becomes

$$-3\begin{vmatrix} 1 & 1 & 4 & -4 \\ 2 & 3 & 9 & 1 \\ -1 & 2 & 8 & 2 \\ -1 & 0 & 2 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & 1 & 4 & -4 \\ 0 & 1 & 1 & 9 \\ 0 & 3 & 12 & -2 \\ 0 & 1 & 6 & -3 \end{vmatrix} = -3\begin{vmatrix} 1 & 1 & 9 \\ 3 & 12 & -2 \\ 1 & 6 & -3 \end{vmatrix} = -3\begin{vmatrix} 1 & 1 & 9 \\ 0 & 9 & -29 \\ 0 & 5 & -12 \end{vmatrix} =$$

$$-3(9 \cdot (-12) + 5 \cdot 29) = -3(-108 + 145) = -3 \cdot 37 = \boxed{-111}$$

3. Suppose A, B and C are 5×5 matrices for which $\det(A) = \frac{1}{2}$, $\det(B) = -5$ and $\det(C) = 7$.
Compute the determinant of the matrix $C(3A)^{-1}B^{-1}$.

$$\det(C(3A)^{-1}B^{-1}) = \det(C)\det((3A)^{-1})\det(B^{-1}) = 7\frac{1}{\det(3A)}\frac{1}{\det(B)} = 7\frac{1}{3^5\det(A)}\frac{1}{\det(B)} =$$

$$\frac{7}{3^5 \cdot \frac{1}{2} \cdot (-5)} = \boxed{-\frac{14}{1215}}$$

4. A square matrix A is called an *orthogonal matrix* if $AA^T = I$.
If A is orthogonal, what are the possible values for $\det(A)$?

Suppose A is orthogonal, so $AA^T = I$. Taking the determinant of both sides we get
$\det(AA^T) = \det(I)$
$\det(A)\det(A^T) = 1$
$\det(A)\det(A) = 1$
$(\det(A))^2 = 1$

Since the square of $\det(A)$ is 1, it follows that $\boxed{\det(A) \text{ equals either } 1 \text{ or } -1.}$

5. Suppose A is a square matrix, and $A^2 = A$.
What are the possible values for $\det(A)$? Explain.

$A^2 = A$
$\det(A^2) = \det(A)$
$\det(AA) = \det(A)$
$\det(A)\det(A) = \det(A)$
$\det(A)^2 - \det(A) = 0$
$\det(A)(\det(A) - 1) = 0$

From this equation, it follows that $\boxed{\text{either } \det(A) = 0 \text{ or } \det(A) = 1.}$

6. Suppose A is an $n \times n$ matrix, and $\det(A^{-1}) = \dfrac{1}{3}$, and $\det(2A) = 384$.

Find the dimension n of A. Explain your reasoning.

From $\det(A^{-1}) = \dfrac{1}{3}$, it follows that $\det(A) = 3$, so

$\det(2A) = 384$
$2^n \det(A) = 384$
$2^n \cdot 3 = 384$
$2^n = 128$
$n = 7$

Thus $\boxed{A \text{ is a } 7 \times 7 \text{ matrix.}}$

7. Find all values of a that make $\begin{bmatrix} a & a & 0 \\ a^2 & 2 & a \\ 0 & a & a \end{bmatrix}$ singular.

$$\begin{vmatrix} a & a & 0 \\ a^2 & 2 & a \\ 0 & a & a \end{vmatrix} = a\begin{vmatrix} 2 & a \\ a & a \end{vmatrix} - a\begin{vmatrix} a^2 & a \\ 0 & a \end{vmatrix} = a(2a-a^2)-a(a^3-0) = 2a^2-a^3-a^4 = -a^2(a^2+a-2) = a^2(a-1)(a+2)$$

From this you can see that $\boxed{\text{the matrix will be singular for } a=0,\ a=1 \text{ and } a=-2.}$

8. Consider the matrix equation $\begin{bmatrix} 12 & -15 & 0 & 5 & -8 \\ -4 & 0 & 1 & 0 & 10 \\ 20 & 3 & 5 & -1 & 0 \\ 14 & -12 & 9 & 4 & 12 \\ 0 & 9 & 10 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Explain how you know this has more than one solution without making any explicit calculations.

This is an equation of form $A\mathbf{x} = \mathbf{0}$. Notice that the second column of the matrix A is a multiple of -3 times the fourth column. Thus $\det(A) = 0$. Hence by the equivalent conditions for a nonsingular matrix (page 144), it is false that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Thus $A\mathbf{x} = \mathbf{0}$ has the trivial solution $\mathbf{x} = \mathbf{0}$, and some other solution as well. That is, it has more than one solution.

The problems on this page concern the matrix $A = \begin{bmatrix} 37 & 105 \\ -14 & -40 \end{bmatrix}$.

9. Find the eigenvalues of A.

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 37 & -105 \\ 14 & \lambda + 40 \end{vmatrix} = (\lambda - 37)(\lambda + 40) + 14 \cdot 105 = \lambda^2 + 3\lambda - 1480 + 1470 = \lambda^2 + 3\lambda - 10 =$
$(\lambda - 2)(\lambda + 5) = 0$

From this you can see that the eigenvalues are $\boxed{\lambda = 2 \text{ and } \lambda = -5.}$

10. For each eigenvalue from Question 9, find the corresponding eigenvectors.

Eigenvectors for $\lambda = 2$:
$(\lambda I - A)\mathbf{x} = \mathbf{0}$
$(2I - A)\mathbf{x} = \mathbf{0}$
$\begin{bmatrix} -35 & -105 \\ 14 & 42 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -35 & -105 & 0 \\ 14 & 42 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $x = -3y$, i.e. $x = -3t,\ y = t$ so $\boxed{\text{the eigenvectors for } \lambda = 2 \text{ are } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}}$

Eigenvectors for $\lambda = -5$:
$(\lambda I - A)\mathbf{x} = \mathbf{0}$
$(-5I - A)\mathbf{x} = \mathbf{0}$
$\begin{bmatrix} -42 & -105 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -42 & -105 & 0 \\ 14 & 35 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & 0 \\ 2 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $x = -\frac{5}{2}y$, i.e. $x = -\frac{5}{2}t,\ y = t$ so $\boxed{\text{the eigenvectors for } \lambda = -5 \text{ are } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}}$

Note: By scaling the eigenvector $\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$ by a factor of 2, we can say that eigenvectors for $\lambda = -5$ are the scalar multiples of $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$