Linear Algebra | Test #2 | March 26, 2007

Name: ______________________ Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is **not** allowed on this test.

1. (5 points) Consider the vectors $\mathbf{u} = (8, -2, 0, 1)$, $\mathbf{v} = (3, -2, 1, 1)$ and $\mathbf{w} = (1, 4, 4, 1)$. in $\mathbb{R}^4$. Find $\mathbf{x}$, given that $\mathbf{w} - 2\mathbf{x} = \mathbf{u} - 3\mathbf{v}$.

$$\begin{aligned}
\mathbf{w} - 2\mathbf{x} &= \mathbf{u} - 3\mathbf{v} \\
-2\mathbf{x} &= \mathbf{u} - 3\mathbf{v} - \mathbf{w} \\
\mathbf{x} &= -\frac{1}{2}(\mathbf{u} - 3\mathbf{v} - \mathbf{w}) \\
\mathbf{x} &= -\frac{1}{2}[(8, -2, 0, 1) - 3(3, -2, 1, 1) - (1, 4, 4, 1)] \\
\mathbf{x} &= -\frac{1}{2}(-2, 0, -7, -3) \\
\mathbf{x} &= (1, 0, \frac{7}{2}, \frac{3}{2})
\end{aligned}$$

2. (5 points) State what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\}$ to be linearly independent.

   The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\}$ is linearly independent if the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_2\mathbf{v}_3, \ldots + c_k\mathbf{v}_k = \mathbf{0}$ has no non-trivial solutions. (That is if it has only the trivial solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $\ldots$ $c_k = 0$.)

3. (10 points) Say if each of the following sets is a vector space. (A "Yes" or "No" answer will suffice.)

   (a) $\{(x, y) \in \mathbb{R}^2 \ : \ x - y = 0\}$ .... YES

   (b) $\left\{\begin{bmatrix} x & x+1 \\ x & x \end{bmatrix} \ : \ x \in \mathbb{R}\right\}$ .... NO

   (c) $\left\{\begin{bmatrix} 0 & x \\ 2x & 0 \end{bmatrix} \ : \ x \in \mathbb{R}\right\}$ .... YES

   (d) $\{f \in C(-\infty, \infty) \ : \ f(3) = 0\}$ .... YES

   (e) P_9 .... YES

   (f) $M_{9,3}$ .... YES

   (g) $\{(x, y) \in \mathbb{R}^2 \ : \ x \geq 0, y \geq 0\}$ .... NO

   (h) $\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 > 0\}$ .... NO

   (i) $\{(x, y, z) \in \mathbb{R}^3 \ : \ x = 0\}$ .... YES

   (j) $\{(0, 0, 0)\}$ .... YES

4. (20 points) The problems on this page concern the matrix $A = \begin{bmatrix} -7 & -5 \\ 10 & 8 \end{bmatrix}$.

(a) Find the eigenvalues of A.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+7 & 5 \\ -10 & \lambda-8 \end{vmatrix} = (\lambda+7)(\lambda-8)+50 = \lambda^2 - \lambda - 6 =)\ (\lambda-3)(\lambda+2) = 0$$

From this you can see that the eigenvalues are $\boxed{\lambda = 3 \text{ and } \lambda = -2.}$

(b) For each eigenvalue from part (a), find the corresponding eigenvectors.

Eigenvectors for $\lambda = 3$:
$(\lambda I - A)\mathbf{x} = \mathbf{0}$
$(3I - A)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 5 & 0 \\ -10 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $2x = -y$, i.e. $x = t,\ y = -2t$ so $\boxed{\text{the eigenvectors for } \lambda = 3 \text{ are } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

Note: If you made y the parameter, your solution would be $x = -\frac{1}{2}t,\ y = t$ and your eigenvectors would be $\begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$. This is correct too, since $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Eigenvectors for $\lambda = -2$:
$(\lambda I - A)\mathbf{x} = \mathbf{0}$
$(-2I - A)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 & 0 \\ -10 & -10 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $x = -y$, i.e. $x = -t,\ y = t$ so $\boxed{\text{the eigenvectors for } \lambda = -2 \text{ are } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

5. (10 points) Show that $W = \{(a, b, a+2b) \ : \ a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

In words, W is the set of all vectors in $\mathbb{R}^3$ whose third component equals the first component plus twice the second.

(a) Suppose that $\mathbf{x}$ and $\mathbf{y}$ are matrices in the set W.
This means $\mathbf{x} = (x, y, x+2y)$ and $\mathbf{y} = (x', y', x'+2y')$, for some $x, y, x', y' \in \mathbb{R}$.
Then $\mathbf{x}+\mathbf{y} = (x, y, x+2y) + (x', y', x'+2y') = (x+x', y+y', x+2y+x'+2y') =$ $(x+x', y+y', (x+x')+2(y+y'))$, and this is in W because it has the form $(a, b, a+2b)$ required for a vector in W.
Therefore W is closed under addition.

(b) Suppose that $\mathbf{x} \in W$ and $c \in \mathbb{R}$.
This means $\mathbf{x} = (x, y, x+2y)$ for some $x, y \in \mathbb{R}$.
Then $c\mathbf{x} = c(x, y, x+2y) = (cx, cy, cx+2cy)$ is in W because it has the form required for a vector in W.
Therefore W is closed under scalar multiplication.

Parts (a) and (b) above show that W is closed under addition and scalar multiplication, so W is a subspace.

6. (10 points) Suppose A is a fixed 2×2 matrix. Show that the set $W = \{X \ : \ AX = XA\}$ is a subspace of $M_{2,2}$.

(a) Suppose that B and C are matrices in the set W.
This means $AB = BA$ and $AC = CA$.
Then $A(B+C) = AB + AC = BA + CA = (B+C)A$.
And $A(B+C) = (B+C)A$ means that $B+C \in W$.
Therefore W is closed under addition.

(b) Suppose that $B \in W$ and $c \in \mathbb{R}$.
The fact that $B \in W$ means $AB = BA$.
Observe that $A(cB) = c(AB) = c(BA) = (cB)A$.
And $A(cB) = (cB)A$ means $cB \in W$.
Therefore W is closed under scalar multiplication.

Parts (a) and (b) above show that W is closed under addition and scalar multiplication, so W is a subspace.

7. (10 points) Suppose W is the set of all matrices in $M_{2,2}$ that have have determinant equal to 0. Explain why W is not a subspace of $M_{2,2}$.

Observe that the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both have determinant 0, so they are in W.

But their sum $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has determinant $1 \neq 0$, so it is not in W.

Thus the set W is not closed under addition, so it is not a subspace of $M_{2,2}$.

8. (10 points) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three vectors in a vector space V. Without knowing any further information, is it possible to say whether or not the set $\{\mathbf{v}-\mathbf{u}, \ \ \mathbf{w}-\mathbf{v}, \ \ \mathbf{u}-\mathbf{w}\}$ is linearly independent or dependent?

Notice that $1(\mathbf{v}-\mathbf{u}) + 1(\mathbf{w}-\mathbf{v}) + 1(\mathbf{u}-\mathbf{w}) = \mathbf{0}$, so it follows the set S is $\boxed{\text{linearly dependent.}}$

Note: One common mistake was to take the equation $c_1(\mathbf{v}-\mathbf{u}) + c_2(\mathbf{w}-\mathbf{v}) + c_3(\mathbf{u}-\mathbf{w}) = \mathbf{0}$, regroup it as $(c_3-c_1)\mathbf{u} + (c_1-c_2)\mathbf{v} + (c_2-c_3)\mathbf{w} = \mathbf{0}$, get $c_3 - c_1 = 0$, $c_1 - c_2 = 0$, $c_2 - c_3 = 0$, and solve the resulting system.

But there is a problem with this approach. Since we are not given that the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we can't deduce that $c_3 - c_1 = 0$, $c_1 - c_2 = 0$, $c_2 - c_3 = 0$.

9. (10 points) Is the set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ linearly independent or dependent?

We need to see whether the equation $c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has any nontrivial solutions.

This equation gives rise to the following system.

$$\begin{cases} c_1 + 2c_2 + c_3 = 0 \\ c_1 + c_2 + c_3 = 0 \\ 3c_2 + c_3 = 0 \\ c_1 + c_2 + c_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So there's only the trivial solution $c_1 = 0,\ c_2 = 0,\ c_3 = 0$, so the set is **LINEARLY INDEPENDENT.**

10. (10 points) Does the set $\{1 + x,\ \ x + x^2,\ \ x^2 + x^3,\ \ 1 + x^3\}$ span P_3?

Given an arbitrary polynomial $a + bx + cx^2 + dx^3$, we want to know if we can always find values for c_1, c_2, c_3, c_4 satisfying $c_1(1 + x) + c_2(x + x^2) + c_3(x^2 + x^3) + c_4(1 + x^3) = a + bx + cx^2 + dx^3$.

Combining, we get $(c_1 + c_4) + (c_1 + c_2)x + (c_2 + c_3)x^2 + (c_3 + c_4)x^3 = a + bx + cx^2 + dx^3$, which leads to the following system.

$$\begin{cases} c_1 + c_4 = a \\ c_1 + c_2 = b \\ c_2 + c_3 = c \\ c_3 + c_4 = d \end{cases}$$

Solving:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & a \\ 1 & 1 & 0 & 0 & b \\ 0 & 1 & 1 & 0 & c \\ 0 & 0 & 1 & 1 & d \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & a \\ 0 & 1 & 0 & -1 & b - a \\ 0 & 1 & 1 & 0 & c \\ 0 & 0 & 1 & 1 & d \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & a \\ 0 & 1 & 0 & -1 & b - a \\ 0 & 0 & 1 & 1 & c - b + a \\ 0 & 0 & 1 & 1 & d \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & a \\ 0 & 1 & 0 & -1 & b - a \\ 0 & 0 & 1 & 1 & c - b + a \\ 0 & 0 & 0 & 0 & d - c + b - a \end{bmatrix}$$

Looking at the last row, you can see the system has no solutions for some values of a, b c and d.
In particular if $a = 0$, $b = 0$, $c = 0$ and $d = 1$, there will be no solution.

Thus the polynomial $0 + 0x + 0x^2 + x^3$ cannot be written as a linear combination of elements in the set, so **the set does not span P_3.**