Name: ______________________ Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is **not** allowed on this test.

1. (25 points) For this problem, $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and $D = \begin{bmatrix} -2 & 0 \end{bmatrix}$.

   Preform the indicated operations or state that they are not possible.

   (a) $BA = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -7 \\ -3 & -1 & 7 \end{bmatrix}$

   (b) $A^T C = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 22 \end{bmatrix}$

   (c) $B^{-1} = \frac{1}{2-2} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ Operation can't be done because it involves division by 0. No inverse exists.

   (d) $CD = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -8 & 0 \end{bmatrix}$

   (e) Solve the equation $X - 3B + 2I_2 = O$ for X.

$$
\begin{aligned}
X - 3B + 2I_2 &= O \\
X &= 3B - 2I_2 \\
X &= 3\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
X &= \begin{bmatrix} 6 & -3 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \\
X &= \begin{bmatrix} 4 & -3 \\ -6 & 1 \end{bmatrix}
\end{aligned}
$$

2. (15 points) Suppose A, B and C are invertible matrices. Solve the equation $AXC = CB$ for X.

$$\begin{aligned} AXC &= CB \\ A^{-1}AXC &= A^{-1}CB \\ IXC &= A^{-1}CB \\ XC &= A^{-1}CB \\ XCC^{-1} &= A^{-1}CBC^{-1} \\ XI &= A^{-1}CBC^{-1} \\ X &= A^{-1}CBC^{-1} \end{aligned}$$

3. (15 points) Find the inverse of the matrix $A = \begin{bmatrix} 3 & 5 & 5 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 5 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 3 & 5 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad R_2 - 3R_1 \to R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{c} R_1 + 2R_2 \to R_1 \\ R_3 + R_2 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right] \quad R_2 + R_3 \to R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 0 \\ 0 & -1 & 0 & 2 & -6 & 1 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right] \quad -R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 0 \\ 0 & 1 & 0 & -2 & 6 & -1 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

Thus $\boxed{A^{-1} = \begin{bmatrix} 2 & -5 & 0 \\ -2 & 6 & -1 \\ 1 & -3 & 1 \end{bmatrix}.}$

4. (15 points) Find A, given that $(2A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned} (2A)^{-1} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \left((2A)^{-1}\right)^{-1} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \\ 2A &= \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ 2A &= -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ A &= -\frac{1}{4} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ A &= \begin{bmatrix} -1 & 1/2 \\ 3/4 & -1/4 \end{bmatrix} \end{aligned}$$

5. (15 points) Factor the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ into a product of elementary matrices.

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} R_2 - R_1 \rightarrow R_2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tfrac{1}{2}R_2 \rightarrow R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This sequence of row reductions shows $\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1}$

So $\boxed{A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}$

Note: There are other correct answers. Depending on the sequence of row operations you used, you may have obtained a different (but still correct) factoring of A into elementary matrices.

6. (15 points) Find an LU factorization of the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix} R_2 - 2R_1 \rightarrow R_2 \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix} R_3 + R_1 \rightarrow R_3 \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} R_3 - R_2 \rightarrow R_3 \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

This sequence of row operations shows $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

Thus $A = \left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \right) \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

So $A = \left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

Thus $\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.}$