Linear Algebra | Test #1 | February 19, 2007

Name: ______________________ R. Hammack Score: ________

Directions: This is a take-home test. It is due at the beginning of class on Wednesday, February 21. Please answer all questions in the space provided. Consider working the problems on scratch paper, then rewriting them neatly on the test. Additional copies of this test can be downloaded from my web page if needed.

For this test, you may discuss the problems among yourselves and share ideas, but the work you turn in must be your own (not copied). At the end of your solution to each problem, please list who (if anyone) you talked to about that problem, plus any additional information you want me to know (e.g. that you *gave* more help than you *received*, or vice versa, etc.).

- Please clearly indicate all of your row operations (e.g. $R_2 + 3R_4 \rightarrow R_2$, etc).
- Indicate your solution clearly by putting it in a $\boxed{\text{box}}$.
- Constants that are not integers should be expressed as fractions.
- You may consult your text and notes, but **no** other source.
- In order to get full credit, you must show or explain all of your work.

1. (20 points) Find the determinants. If you need more room for (d), reconsider your approach.

(a) $\begin{vmatrix} -5 & -\frac{1}{2} \\ 10 & 1 \end{vmatrix} = (-5)(1) - (10)(-\frac{1}{2}) = \boxed{0}$

(b) $\begin{vmatrix} 5 & 1 & 3 \\ -3 & 4 & 4 \\ 5 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -23 & 4 & -8 \\ -20 & 5 & -10 \end{vmatrix} = 0 - \begin{vmatrix} -23 & -8 \\ -20 & -10 \end{vmatrix} + 0 = \boxed{\text{-70}}$

(Column operations $C_1 - 5C_2 \rightarrow C_1$ and $C_3 - 3C_2 \rightarrow C_3$ followed by expansion along row 1.)

(c) $\begin{vmatrix} 1 & -1 & -2 & 5 \\ 3 & 3 & 6 & 1 \\ 0 & 1 & 3 & 0 \\ -2 & 4 & 8 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 & 5 \\ 0 & 6 & 12 & -14 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 4 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -2 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 6 & 12 & -14 \\ 0 & 2 & 4 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -2 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -6 & -14 \\ 0 & 0 & -2 & 8 \end{vmatrix} =$

$-\begin{vmatrix} 1 & -1 & -2 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -38 \\ 0 & 0 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & -38 \end{vmatrix} = \boxed{76}$

(d) $\begin{vmatrix} 2 & -1 & -2 & 5 & 2 & 1 & 6 \\ 4 & 3 & 4 & 5 & 2 & 1 & 6 \\ 1 & 1 & -3 & 0 & 2 & -1 & 2 \\ 2 & 2 & -6 & 0 & 4 & -2 & 4 \\ 1 & 2 & -6 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & -6 & 1 & 1 & 1 & 1 \end{vmatrix} = \boxed{0}$ (Because row 4 is a multiple of row 3.)

2. (10 points) Solve the system $\left\{\begin{array}{rcrcrcrcr} 2w & - & x & - & y & + & 3z & = & 0 \\ w & + & x & - & 2y & + & z & = & 1 \\ 2w & + & x & + & y & - & z & = & 2 \\ 3w & - & x & + & y & + & z & = & 1 \end{array}\right.$

$$\left[\begin{array}{rrrrr} 2 & -1 & -1 & 3 & 0 \\ 1 & 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & -1 & 2 \\ 3 & -1 & 1 & 1 & 1 \end{array}\right] R_2 \leftrightarrow R_1 \left[\begin{array}{rrrrr} 1 & 1 & -2 & 1 & 1 \\ 2 & -1 & -1 & 3 & 0 \\ 2 & 1 & 1 & -1 & 2 \\ 3 & -1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \\ R_4 - 3R_1 \to R_4 \end{array}$$

$$\left[\begin{array}{rrrrr} 1 & 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 1 & -2 \\ 0 & -1 & 5 & -3 & 0 \\ 0 & -4 & 7 & -2 & -2 \end{array}\right] R_2 \leftrightarrow R_3 \left[\begin{array}{rrrrr} 1 & 1 & -2 & 1 & 1 \\ 0 & -1 & 5 & -3 & 0 \\ 0 & -3 & 3 & 1 & -2 \\ 0 & -4 & 7 & -2 & -2 \end{array}\right] -R_2 \to R_2$$

$$\left[\begin{array}{rrrrr} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & -5 & 3 & 0 \\ 0 & -3 & 3 & 1 & -2 \\ 0 & -4 & 7 & -2 & -2 \end{array}\right] \begin{array}{l} R_1 - R_2 \to R_1 \\ R_3 + 3R_2 \to R_3 \\ R_4 + 4R_2 \to R_4 \end{array} \left[\begin{array}{rrrrr} 1 & 0 & 3 & -2 & 1 \\ 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & -12 & 10 & -2 \\ 0 & 0 & -13 & 10 & -2 \end{array}\right] R_3 - R_4 \to R_3$$

$$\left[\begin{array}{rrrrr} 1 & 0 & 3 & -2 & 1 \\ 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -13 & 10 & -2 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \to R_1 \\ R_2 + 5R_3 \to R_2 \\ R_4 + 13R_3 \to R_4 \end{array} \left[\begin{array}{rrrrr} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 & -2 \end{array}\right] \tfrac{1}{10}R_4 \to R_4$$

$$\left[\begin{array}{rrrrr} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{5} \end{array}\right] \begin{array}{l} R_1 + 2R_4 \to R_1 \\ R_2 - 3R_4 \to R_2 \end{array} \left[\begin{array}{rrrrr} 1 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 1 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

SOLUTION: $w = \frac{3}{5}$, $x = \frac{3}{5}$, $y = 0$, $z = -\frac{1}{5}$

3. (10 points) Solve the system $\begin{cases} 2w + x + y + 3z = 1 \\ 2w - 7x + y + 11z = 1 \\ 2w + 3x + y + z = 1 \end{cases}$

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 1 \\ 2 & -7 & 1 & 11 & 1 \\ 2 & 3 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \begin{bmatrix} 2 & 1 & 1 & 3 & 1 \\ 0 & -8 & 0 & 8 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{bmatrix} \begin{matrix} -\frac{1}{8}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \to R_1 \\ R_3 - R_2 \to R_3 \end{matrix} \begin{bmatrix} 2 & 0 & 1 & 4 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tfrac{1}{2}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system $\begin{cases} w + \frac{1}{2}y + 2z = \frac{1}{2} \\ x - z = 0 \end{cases}$

SOLUTION: $\boxed{w = \frac{1}{2} - \frac{1}{2}s - 2t, \quad x = t, \quad y = s, \quad z = t}$

4. (10 points) Solve the system $\begin{cases} x + 4y + 5z = 1 \\ x + 18y + 7z = 2 \\ x - 3y + 4z = 1 \end{cases}$

$$\begin{bmatrix} 1 & 4 & 5 & 1 \\ 1 & 18 & 7 & 2 \\ 1 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 1 \\ 0 & 14 & 2 & 1 \\ 0 & -7 & -1 & 0 \end{bmatrix} \tfrac{1}{2}R_2 \to R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 1 \\ 0 & 7 & 1 & 1/2 \\ 0 & -7 & -1 & 0 \end{bmatrix} R_3 + R_2 \to R_3 \begin{bmatrix} 1 & 4 & 5 & 1 \\ 0 & 7 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

The last row corresponds to the equation $0x + 0y + 0z = 1/2$.
From this we deduce that the system has $\boxed{\text{NO SOLUTIONS.}}$

5. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 1 & 2 & 2 \end{bmatrix}$.

(a) (10 points) Find an LU factorization of A.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Therefore $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus $\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

(b) (10 points) Find the inverse of A. (If it exists.)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 7 & 3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 - 3R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] R_1 - R_3 \to R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] R_1 - 2R_2 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -2 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Therefore $\boxed{A^{-1} = \begin{bmatrix} 8 & -2 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$

6. (6 points) Suppose $A = [\,3\ \ 1\ \ 2\,]$ and $B = [\,1\ \ 1\ \ -2\,]$. Find X if $A^T B - X = 2I_3$.

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} [\,1\ 1\ -2\,] - X = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix} - X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 3 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 3 & -6 \\ 1 & -1 & -2 \\ 2 & 2 & -6 \end{bmatrix}}$$

7. (6 points) If $D = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$, find D^{-2}.

$$D^{-2} = (D^2)^{-1} = \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix}^{-1} = \frac{1}{(-1)\cdot(-2) - 2\cdot(-1)} \begin{bmatrix} -2 & -2 \\ 1 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}}$$

8. (6 points) Suppose A and B are 4×4 matrices, and $\det(A) = 5$, and $\det(B) = -2$. Find $\det(2A^2B^{-3})$.

$$\det(2A^2B^{-3}) = 2^4 \det(A^2B^{-3}) = 16\det(A^2(B^3)^{-1})$$
$$= 16\det(A^2)\det((B^3)^{-1}) = 16\det(AA)\det((B^3)^{-1})$$
$$= 16\det(A)\det(A)\det((B^3)^{-1}) = \frac{16\det(A)\det(A)}{\det(B^3)} =$$
$$\frac{16\det(A)\det(A)}{\det(B)\det(B)\det(B)} = \frac{16\cdot 5\cdot 5}{(-2)(-2)(-2)} = \boxed{-50}$$

9. (6 points) Find an example of a 2×2 matrix A for which $A \neq O$, but $A^2 = O$.

Let $\boxed{A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.}$ Then $A \neq O$, but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

10. (6 points) Suppose A is a 7×7 matrix, and $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$. Is A invertible or singular? Explain.

This is saying that the equation $A\mathbf{x} = \mathbf{0}$ has a *NONTRIVIAL* solution.
By the Equivalent Conditions for a Nonsingular Matrix (page 144),
it follows that A is not invertible, so $\boxed{A \text{ is singular.}}$