Linear Algebra | Quiz for Section 4.5 | November 5, 2009

Name: ________________ R. Hammack Score: ______

Directions: Please answer all questions in the space provided. Use of calculators or any form of electronic communication device is strictly forbidden on this quiz.

1. A matrix A is **symmetric** if $A^T = A$. Find a basis for the vector space of all 3×3 symmetric matrices.

If A is a 3×3 symmetric matrix, then the condition $A^T = A$ means A has form $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$, for $a, b, c, d, e, f \in \mathbb{R}$.

Let $V = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}$ be the set of all symmetric 3×3 matrices (which is easily seen to be a subspace of $M_{3,3}$.) Notice that for any symmetric matrix A we have

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the set $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ spans V.

Also, the set S is linearly independent because if

$$c_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_5\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + c_6\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\begin{bmatrix} c_1 & c_2 & c_3 \\ c_2 & c_4 & c_5 \\ c_3 & c_5 & c_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and consequently $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$.

So we have seen that the set S spans V and is linearly independent. Conclusion:

The set $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ is a basis for V.

You had one of the following problems for your second question:

2. Consider the following subspace of $\mathbb{R}^4$: $W = \{(t, 3s - 4t, t + s, s) \ : \ s \text{ and } t \text{ are real numbers}\}$
Find a basis for W. State the dimension of W.

Note that $(t, 3s - 4t, t + s, s) = s(0, 3, 1, 1) + t(1, -4, 1, 0)$.
This means any vector in W is a linear combination of $(0, 3, 1, 1)$ and $(1, -4, 1, 0)$.
Thus the set $S = \{(0, 3, 1, 1), (1, -4, 1, 0)\}$ spans W.
Moreover, the set S of two vectors is linearly independent because no vector in S is a multiple of the other.
Since S is linearly independent and spans W, it is a basis for W.
Since the basis has two elements, the subspace W has dimension 2.
Conclusion: The set {(0,3,1,1), (1,-4,1,0)} is a basis for W, and W is two-dimensional.

3. Consider the following subspace of $\mathbb{R}^4$: $W = \{(2s - 3t, s, t + s, t) \ : \ s \text{ and } t \text{ are real numbers}\}$
Find a basis for W. State the dimension of W.

Note that $(2s - 3t, s, t + s, t) = s(2, 1, 1, 0) + t(-3, 0, 1, 1)$.
This means any vector in W is a linear combination of $(2, 1, 1, 0)$ and $(-3, 0, 1, 1)$.
Thus the set $S = \{((2, 1, 1, 0), (-3, 0, 1, 1)\}$ spans W.
Moreover, the set S of two vectors is linearly independent because no vector in S is a multiple of the other.
Since S is linearly independent and spans W, it is a basis for W.
Since the basis has two elements, the subspace W has dimension 2.
Conclusion: The set {((2,1,1,0), (-3,0,1,1)} is a basis for W, and W is two-dimensional.