Linear Algebra — Quiz for Section 2.1 — September 10, 2009

Name: ____________________ R. Hammack Score: ______

Directions: Please answer in the space provided. Please show all of your work.
Use of calculators or any form of electronic communication device is strictly forbidden on this quiz.

1. In this problem $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Find conditions on w, x, y and z such that $AB = BA$.

Let's look at the equation $AB = BA$.

$$\begin{aligned} AB &= BA \\ \begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ \begin{bmatrix} w-x & w+x \\ y-z & y+z \end{bmatrix} &= \begin{bmatrix} w+y & x+z \\ -w+y & -x+z \end{bmatrix} \end{aligned}$$

For this to be true, the following four equations must hold.

$$\begin{cases} w-x = w+y \\ w+x = x+z \\ y-z = -w+y \\ y+z = -x+z \end{cases}$$

From this we get the following system.

$$\begin{cases} \quad\; -x - y \quad\quad = 0 \\ w \quad\quad\quad\quad - z = 0 \\ w \quad\quad\quad\quad - z = 0 \\ \quad\;\;\; x + y \quad\quad = 0 \end{cases}$$

Solving this in the usual way, we get:

$$\begin{bmatrix} 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ R_2 \leftrightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} R_4 - R_1 \to R_4 \\ R_3 + R_2 \to R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we see $w = z$ and $x = -y$.

Thus, in order to have $AB = BA$ it must be the case that $w = z$ and $x = -y$.

In other words, A must have form $A = \begin{bmatrix} z & -y \\ y & z \end{bmatrix}$ for numbers $x, y \in \mathbb{R}$.
e.g. $A = \begin{bmatrix} 7 & -4 \\ 4 & 7 \end{bmatrix}$ is one such matrix for which $AB = BA$.