Midterm | Linear Algebra MATH 310 R. Hammack | October 18, 2016

Name: Richard | Score: 100

Answer in the space provided. No calculators. Please put all phones, etc., away. Each problem is 10 points.

1. For this problem, $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and $D = \begin{bmatrix} -2 & 0 \end{bmatrix}$.

   Preform the indicated operations or state that they are not possible.

   (a) $BA = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 1 & -7 \\ -3 & -1 & 7 \end{bmatrix}}$

   (b) $C - \frac{1}{2}D^T = \begin{bmatrix} -2 \\ 4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix}}$

   (c) $B^{-1} =$ $\det(B) = (2)(1) - (-2)(-1) = 0$ so $\boxed{\text{no inverse}}$

   (d) $CD = \begin{bmatrix} -2 \\ 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 0 \\ -8 & 0 \end{bmatrix}}$

   (e) Solve the equation $X - 3B + 2I_2 = O$ for X.

   $X = 3B - 2I = 3\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -3 \\ -6 & 1 \end{bmatrix}}$

2. Suppose A, B and C are invertible matrices. Solve the equation $AXC = CB$ for X.

$$AXC = CB$$
$$A^{-1}AXC = A^{-1}CB$$
$$XC = A^{-1}CB$$
$$XCC^{-1} = A^{-1}CBC^{-1}$$
$$\boxed{X = A^{-1}CBC^{-1}}$$

3. Solve the system $\begin{cases} 4w - 8x - 3y + z = 1 \\ -3w + 6x + 2y + z = 1 \end{cases}$

$$\begin{bmatrix} 4 & -8 & -3 & 1 & 1 \\ -3 & 6 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2 \to R_1} \begin{bmatrix} 1 & -2 & -1 & 2 & 2 \\ -3 & 6 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + 3R_1 \to R_2} \begin{bmatrix} 1 & -2 & -1 & 2 & 2 \\ 0 & 0 & -1 & 7 & 7 \end{bmatrix}$$

$$\xrightarrow{-R_2 \to R_2} \begin{bmatrix} 1 & -2 & -1 & 2 & 2 \\ 0 & 0 & 1 & -7 & -7 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2 \to R_1} \begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & -5 & -5 \\ 0 & 0 & 1 & -7 & -7 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \phantom{-5} \\ w & x & y & z & \phantom{-5} \end{matrix} \end{array}$$ (Reduced row echelon form)

$$\begin{cases} w - 2x \qquad - 5z = -5 \\ \qquad\qquad y - 7z = -7 \end{cases}$$

free variables $x = s$, $z = t$

$$\boxed{\text{solutions} \begin{cases} w = -5 + 2s + 5t \\ x = s \\ y = -7 + 7t \\ z = t \end{cases} \quad \text{where } s, t \in \mathbb{R}}$$

4. Find the inverse of the matrix $A = \begin{bmatrix} 3 & 5 & 5 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 5 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 3 & 5 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \to R_3]{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 0 \\ 0 & -1 & 0 & 2 & -6 & 1 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 0 \\ 0 & 1 & 0 & -2 & 6 & -1 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

$$\boxed{\text{Thus } A^{-1} = \begin{bmatrix} 2 & -5 & 0 \\ -2 & 6 & -1 \\ 1 & -3 & 1 \end{bmatrix}}$$

Check $AA^{-1} = \begin{bmatrix} 3 & 5 & 5 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 & 0 \\ -2 & 6 & -1 \\ 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

5. A square matrix A is called an *orthogonal matrix* if $AA^T = I$.
If A is orthogonal, what are the possible values for $\det(A)$?

$$AA^T = I$$
$$\det(AA^T) = \det(I)$$
$$\det(A)\det(A^T) = 1$$
$$\det(A)\det(A) = 1$$
$$(\det(A))^2 = 1$$

$$\boxed{\det(A) = \pm 1}$$

Recall $\det(A^T) = \det(A)$

6. Find all values of k that make $\begin{bmatrix} 2-k & 1 \\ 6 & 1-k \end{bmatrix}$ singular.

Take the determinant: $(2-k)(1-k) - 6$

$$= 2 - 3k + k^2 - 6$$
$$= k^2 - 3k - 4$$
$$= (k+1)(k-4) = 0$$

$\boxed{\text{Singular when } k=-1 \text{ or } k=4}$

7. Find A, given that $(2A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$((2A)^{-1})^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

$$2A = \frac{1}{(1)(4)-(3)(2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$2A = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A = -\frac{1}{4} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}}$$

Note: Some students used formula $(cA)^{-1} = \frac{1}{c}A^{-1}$, so $(2A)^{-1} = \frac{1}{2}A^{-1}$.
If you did $(2A)^{-1} = 2A^{-1}$, you lost points for incorrect use of the formula.
–RH

8. Suppose $\mathbf{u}_1 = (1,3,5)$, $\mathbf{u}_2 = (2,-1,3)$, $\mathbf{u}_3 = (-3,2,-4)$ and $\mathbf{v} = (-1,7,2)$.
Is $\mathbf{v}$ a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$?

Let's check. If so then there are $x, y, z \in \mathbb{R}$ with

$$x(1,3,5) + y(2,-1,3) + z(-3,2,-4) = (-1,7,2)$$

$$\Rightarrow \begin{cases} x + 2y - 3z = -1 \\ 3x - y + 2z = 7 \\ 5x + 3y - 4z = 2 \end{cases}$$

(solve this)

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 - 5R_1 \to R_3]{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix} \longleftarrow \text{no solution}$$

$\boxed{\text{Therefore } \vec{v} \text{ is \underline{not} a linear combo. of } \vec{u}_1, \vec{u}_2, \vec{u}_3.}$

9. Consider the matrix equation $\begin{bmatrix} 12 & 15 & 5 & 0 \\ -4 & 0 & 0 & 1 \\ 20 & 3 & 1 & 5 \\ 14 & 12 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Explain how you know this has more than one solution without making any explicit calculations.

This has form $A\vec{x} = \vec{0}$, and $\det(A) = 0$ because the $2^{nd}$ column of $A$ is a multiple of the $3^{rd}$ column. A theorem tells us that if $\det(A) = 0$, Then $A\vec{x} = \vec{0}$ <u>does not have</u> only The trivial solution $\vec{x} = \vec{0}$. Thus in addition to the solution $\vec{x} = \vec{0}$, it has other solutions as well.

10. Let A be a fixed $2 \times 2$ matrix. Prove that the set $W = \{ X \in M_{2,2} \ : \ AX = XA \}$ is a subspace of $M_{2,2}$.

We need to show W is closed with respect to ① addition, and ② scalar multiplication.

① Suppose $X, Y \in W$. This means $\boxed{AX = XA}$ and $\boxed{AY = YA}$. Now look at $X+Y$. To show this is in W we must verify that $A(X+Y) = (X+Y)A$. Now, observe that

$$A(X+Y) = AX + AY \quad \text{(distribututive property)}$$
$$= XA + YA \quad \text{(using above boxed equations)}$$
$$= (X+Y)A \quad \text{(distribututive property)}$$

Therefore we see that indeed $A(X+Y) = (X+Y)A$, which means $X+Y \in W$. Thus W is closed under addition.

② Suppose $X \in W$ and $c \in \mathbb{R}$. Because $X \in W$, we know $\boxed{AX = XA}$.

Now look at $cX$. We need to show that $cX$ is in W. Notice that

$$A(cX) = cAX \quad \text{(property of matrix mult.)}$$
$$= cXA \quad \text{(Using boxed equation above)}$$
$$= (cX)A \quad \text{(property of matrix mult.)}$$

Because $A(cX) = (cX)A$ we conclude that $cX \in W$. Therefore W is closed with respect to scalar multiplication.

Items ① and ② above prove that W is a subspace of $M_{2,2}$.