Name: ______________ R. Hammack Score: ______

1. Suppose $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 2 & 1 \end{bmatrix}$.

(a) Find $\mathbf{x}$ if $2\mathbf{u} - 2\mathbf{x} = \mathbf{v}$. $\mathbf{x} = \begin{bmatrix} 1/2 \\ -3 \\ -3 \end{bmatrix}$ (Work omitted. You would have to show it to get full credit.)

(b) $A\mathbf{u} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 11 & 5 \\ 2 & 14 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 11 & 2 & 4 \\ 13 & 6 & -4 \\ 7 & -1 & 8 \end{bmatrix}$

(e) $A + B^T = \begin{bmatrix} 3 & -2 & 6 \\ 6 & 6 & 1 \end{bmatrix}$

(f) Give a basis for the column space of A.

You can see that the first two columns of A are linearly independent. Thus the span of these two columns is a two-dimensional subspace of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is two-dimensional it follows that the first two columns of A span $\mathbb{R}^2$, so they form a basis for $\mathbb{R}^2$. Consequently, these first two vectors form a basis for the column space.

$B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ (*There are many other correct answers.*)

(g) $\text{nullity}(A) =$

Since $3 = \text{rank}(A) + \text{nullity}(A)$, and $\text{rank}(A) = 2$ from part (f) above, it follows that $\text{nullity}(A) = 1$.

2. Solve the system: $\left\{\begin{array}{rcr} w \quad\; + 2y + \;\, z & = & 1 \\ -2w + x + \;\, y - \;\, z & = & -1 \\ w + x + 7y + 3z & = & 2 \end{array}\right.$

$$\left[\begin{array}{rrrrr} 1 & 0 & 2 & 1 & 1 \\ -2 & 1 & 1 & -1 & -1 \\ 1 & 1 & 7 & 3 & 2 \end{array}\right] \to \left[\begin{array}{rrrrr} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 5 & 1 & 1 \\ 0 & 1 & 5 & 2 & 1 \end{array}\right] \to \left[\begin{array}{rrrrr} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 5 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrrrr} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: $\left\{\begin{array}{rcr} w & = & 1-2t \\ x & = & 1-5t \\ y & = & t \\ z & = & 0 \end{array}\right.$

3. Solve the system: $\left\{\begin{array}{rcr} x + y + z & = & 2 \\ x - y - z & = & 0 \\ x + y - z & = & -1 \end{array}\right.$

$$\left[\begin{array}{rrrr} 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & -2 & -3 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

Solution: $\left\{\begin{array}{rcr} x & = & 1 \\ y & = & -1/2 \\ z & = & 3/2 \end{array}\right.$

4. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

5. $$\begin{vmatrix} 3 & 0 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 3(3-6) + 6(3-3) = -9$$

6. Suppose linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.
Find the standard matrix for T.

Note: $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Standard matrix is $A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \; T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right] = \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}$

7. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 1 & 2 & 7 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 5 & 1 \end{bmatrix}$.

(a) Find a basis for the row space of A.

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 1 & 2 & 7 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 2 & 4 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = \{[1\ 0\ 3\ 0],\ [0\ 1\ 2\ 0],\ [0\ 0\ 0\ 1]\}$

(b) Find a basis for the nullspace of A.

From the row reduction above, we can see that the nullspace of A is $\left\{ \begin{bmatrix} -3t \\ -2t \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right)$.

Thus a basis for the nullspace is $B = \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

8. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$, is a linear transformation defined as $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} y - x \\ x + y + z \end{bmatrix}$.

Suppose $S : \mathbb{R}^2 \to \mathbb{R}^3$, is a linear transformation defined as $S\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x - y \\ x \\ y \end{bmatrix}$.

Find the standard matrix for $S \circ T$.

Matrix for T is $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Matrix for S is $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Matrix for $S \circ T$ is $BA = \begin{bmatrix} -2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

9. Suppose a matrix A satisfies $P^{-1}AP = D$, where $P = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
List the eigenvalues of A, and for each eigenvalue, give a basis for its eigenspace.
(This can be done without computations.)

You can read off the eigenvectors from D and the eigenvectors from P.

Eigenvalue $\lambda = 3$ has an eigenspace with basis $\left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigenvalue $\lambda = 4$ has an eigenspace with basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

10. Decide if the polynomials $\{2 - x,\ \ 2x - x^2,\ \ 6 - 5x + x^2\}$ in P_2 are linearly independent or linearly dependent.

By inspection, you can see that $-3(2 - x) + 1(2x - x^2) + 1(6 - 5x + x^2) = 0$, so the polynomials are linearly dependent.

11. Suppose A is an invertible matrix. Prove that if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eivenvector $\mathbf{x}$.

Proof. Suppose λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$.
This means $A\mathbf{x} = \lambda\mathbf{x}$.
Now multiply both sides of this equation with A^{-1}.

$A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x})$

$(A^{-1}A\mathbf{x}) = \lambda A^{-1}\mathbf{x}$

$I\mathbf{x} = \lambda A^{-1}\mathbf{x}$

$\mathbf{x} = \lambda A^{-1}\mathbf{x}$

$\frac{1}{\lambda}(\mathbf{x}) = \frac{1}{\lambda}(\lambda A^{-1}\mathbf{x})$

$\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$

Now observe that the equation $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ means that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eivenvector $\mathbf{x}$. The proof is complete.

12. Suppose $B = \{1, x, e^x, xe^x\}$ is the basis for a subspace W of the space of continuous functions $C(-\infty, \infty)$, and $T : W \to W$ is the linear transformation defined as $T(f) = D_x[f]$ (i.e. $T(f)$ equals the derivative of f).

(a) Find the matrix for T relative to the basis B.

$$A = [\,[T(1)]_B\ [T(x)]_B\ [T(e^x)]_B\ [T(xe^x)]_B]$$

$$= [\,[D_x(1)]_B\ [D_x(x)]_B\ [D_x(e^x)]_B\ [D_x(xe^x)]_B]$$

$$= [\,[0]_B\ [1]_B\ [e^x]_B\ [e^x + xe^x]_B]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find the kernel of T

$\ker(T) = \{a + 0x + 0e^x + 0xe^x \ : \ a \in \mathbb{R}\}$
(since only constant functions have derivatives that are 0.)

(c) Find the rank of T

From part (b) above, we can see that $\text{nullity}(T) = 1$.
Since $\dim(W) = \text{nullity}(T) + \text{rank}(T)$, we get $4 = \text{nullity}(T) + \text{rank}(T)$ or $4 = 1 + \text{rank}(T)$.
Thus the rank of T is 3.

13. Suppose A is a fixed 2×2 matrix. Prove that the set $W = \{X \in M_{2,2} \ : \ XA = AX\}$ is a subspace of $M_{2,2}$.

(1) Suppose matrices B and C are in W.
This means $BA = AB$ and $CA = AC$.
Then $(B + C)A = BA + CA = AB + AC = A(B + C)$.
So $(B + C)A = A(B + C)$, which means $B + C$ is in W.
Thus W is closed under addition.

(2) Suppose matrix B is in W and $c \in \mathbb{R}$.
Since B is in W, we have $BA = AB$.
Now notice that $(cB)A = c(BA) = c(AB) = A(cB)$.
So $(cB)A = A(cB)$, which means cB is in W.
Thus W is closed under scalar multiplication.

Considerations (1) and (2) above show that W is a subspace of $M_{2,2}$.

14. This problem concerns the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

(a) Find all eigenvalues for A.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Thus the only eigenvalue is $\lambda = 1$ (multiplicity 2).

(b) Find all eigenspaces for A.

$$N(1I - A) = N\left(\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 \\ t \end{bmatrix} : t \in \mathbb{R}\right\} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Thus $\text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ is the only eigenspace.

(c) Is A diagonalizable? Explain.

No. All eigenvectors lie on the one-dimensional subspace $\text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

There does not exist a basis of $\mathbb{R}^2$ consisting of eigenvectors.

Thus the given matrix is not diagonalizable.