MATH 310, Section 6.2 Solutions

4. $T : P_2 \to P_1$, $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$. $\boxed{\ker(T) = \{a_0 + 0x + 0x^2 : a_0 \in \mathbb{R}\}.}$

18. $T : \mathbb{R}^4 \to \mathbb{R}^2$ is defined as $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

    A is already in reduced row echelon form, so we can just read off the information.

    (a) $\ker(T) = N(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right)$

    (b) nullity(T)=2.

    (c) range(T) = col(A) = $\mathbb{R}^2$.

    (d) rank(T)=2.

26. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ has rank 1.

    By Theorem 6.5, rank(T) + nullity(T) = 3, hence $\boxed{\text{nullity}(T) = 2}$. This means the kernel of T is two-dimensional, so it is a plane. Since the rank is 1, the range is one-dimensional, so it is a line.

38. (a) $M_{2,3}$ is isomorphic to $\mathbb{R}^6$.

    (b) P_6 is isomorphic to $\mathbb{R}^7$ but **not** to $\mathbb{R}^6$.

    (c) $C[0,6]$ is infinite-dimensional so it is **not** isomorphic to $\mathbb{R}^6$.

    (d) $M_{6,1}$ is isomorphic to $\mathbb{R}^6$.

    (e) P_5 is isomorphic to $\mathbb{R}^6$.

    (f) $\{(x_1, x_2, x_3, 0, x_4, x_5, x_6) : x_i \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}^6$.

46. Verify that $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$ defines a linear transformation T that is one-to-one and onto.

    Doing one row operation, $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 4 & 1 \end{bmatrix}$, and you can see that $\det(A) = -24 \neq 0$.
    Now consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(\mathbf{x}) = A\mathbf{x}$.

    To show T is one-to-one, we must show that the preimage of any vector $\mathbf{b} \in \mathbb{R}^3$ consists of a single vector. This preimage consists of all vectors $\mathbf{x} \in \mathbb{R}^3$ for which $T(\mathbf{x}) = \mathbf{b}$, that is for which $A\mathbf{x} = \mathbf{b}$. By the Equivalent Conditions for a Square Matrix (Page 239), parts 2 and 5 show that $A\mathbf{x} = \mathbf{b}$ has exactly one solution. Thus the preimage of $\mathbf{b}$ consists of a single vector, so T is one-to-one.

    To show that T is onto, we must show that any $\mathbf{b} \in \mathbb{R}^3$ has a preimage in $\mathbb{R}^3$. But that follows from the fact mentioned above, that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ has exactly one solution, so this solution is a preimage of $\mathbf{b}$.