MATH 310, Section 6.1 Solutions

4. $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$

   (a) $T(-4,5,1) = (2\cdot(-4)+5, 2\cdot 5 - 3\cdot(-4), -4-1) = \boxed{(-3,22,-5)}$

   (b) We need to find all the points (v_1, v_2, v_3) for which $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3) = (4, 1, -1)$.

   This gives the following system: $\begin{cases} 2v_1 + v_2 = 4 \\ -3v_1 + 2v_2 = 1 \\ v_1 - v_3 = -1 \end{cases}$

   $$\begin{bmatrix} 2 & 1 & 0 & 4 \\ -3 & 2 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 \\ -3 & 2 & 0 & 1 \\ 2 & 1 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -3 & -2 \\ 0 & 1 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 2 & -3 & -2 \end{bmatrix} \to$$

   $$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & -7 & -14 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{cases} v_1 = 1 \\ v_2 = 2 \\ v_3 = 2 \end{cases}$$

   $\boxed{\text{Thus the preimage of } (4,1,-2) \text{ is } (1,2,2).}$

8. Is $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined as $T(x,y,z) = (x+y, x-y, z)$ a linear transformation?
   Notice that:

   (a) $T((x,y,z) + (x' + y' + z')) = T(x+x', y+y', z+z') = (x+x'+y+y', x+x'-(y+y'), z+z')$
   $= (x+y, x-y, z) + (x'+y', x'-y', z') = T(x,y,z) + T(x', y', z')$

   (b) $T(c(x,y,z)) = T(cx, cy, cz) = (cx+cy, cx-cy, cz) = c(x+y, x-y, z) = cT(x,y,z)$

   Therefore $\boxed{T \text{ is linear.}}$

18. $T(-2,4,-1) = T(-2(1,0,0) + 4(0,1,0) - (0,0,1)) = T(-2(1,0,0)) + T(4(0,1,0)) - T(0,0,1) = -2T(1,0,0) + 4T(0,1,0)) - T(0,0,1) = -2(2,4,-1) + 4(1,3,-2) - (0-2,2) = \boxed{(0,6,-8)}$

38. $$T\left(\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) =$$
   $$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) + 4T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) =$$
   $$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & -1 \\ 7 & 4 \end{bmatrix}}$$

50. Prove that the set of all fixed points of a linear transformation $T : V \to V$ is a subspace of V.
    Proof. The set of fixed points is $W = \{\mathbf{u} \in V \ : \ T(\mathbf{u}) = \mathbf{u}\}$.

    Note that W is closed under addition: Suppose $\mathbf{u}, \mathbf{v} \in W$.
    This means $T(\mathbf{u}) = \mathbf{u}$ and $T(\mathbf{v}) = \mathbf{v}$.
    Then $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{u}+\mathbf{v}$, so $\mathbf{u}+\mathbf{v}$ is a fixed point, so it is in W.

    Next observe that W is closed under scalar multiplication:
    Suppose $\mathbf{u} \in W$ and $c \in \mathbb{R}$.
    Then $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{u}$.
    Thus $c\mathbf{u}$ is a fixed point, so it is in W.

    It follows from Theorem 4.5 that the set of fixed points is a subspace.