**MATH 310, Section 4.6 Solutions**

2. $A = [1\ 2\ 3]$

   (a) The rank of $A$ is 1

   (b) A basis for the row space is $\{(1,2,3)\}$

   (c) A basis for the column space is $\{[1]\}$

8. $$A = \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 1 & 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -1 \\ 7 & 14 & -6 & -3 \\ 2 & 4 & -3 & -6 \\ -2 & -4 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

   $$\to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ From this, you can read off all the answers:

   (a) The rank of $A$ is 2.

   (b) Basis for the row space is $\{(1,2,0,3), (0,0,1,4)\}$

   (c) Basis for the column space is $\left\{ \begin{bmatrix} 2 \\ 7 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 1 \\ -1 \end{bmatrix} \right\}$

12. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $S = \{(2,5,-3,-2), (-2,-3,2,-5), (1,3,-2,2), (-1,-5,3,5)\}$.

    For this, we find a basis of the row space of $A = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 2 \\ -2 & -3 & 2 & -5 \\ 2 & 5 & -3 & -2 \\ -1 & -5 & 3 & 5 \end{bmatrix}$

    $$\to \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 6 \\ 0 & 3 & -2 & -1 \\ 0 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -16 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & -1 & 19 \end{bmatrix}$$

    $$\to \begin{bmatrix} 1 & 0 & 1 & -16 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

    From this, you can see that a basis for the row space (hence for the subspace spanned by the rows) is $\boxed{S = \{(1,0,0,3), (0,1,0,-13), (0,0,1,-19)\}}$.

18. Find a basis for and the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}$.

    Working this out with Gauss-Jordan, $\begin{bmatrix} 3 & -6 & 21 & 0 \\ -2 & 4 & -14 & 0 \\ 1 & -2 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & 0 \\ -2 & 4 & -14 & 0 \\ 3 & -6 & 21 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

    Thus, the solutions are $\mathbf{x} = \begin{bmatrix} 2s - 7t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$

Thus, $\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the solution space}}$, and $\boxed{\text{the dimension of the solution space is 2.}}$

19. Determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

In other words, determine if $x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has any solutions.

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1/2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$$

As you can now see, there are no solutions to this system, so $\boxed{\mathbf{b} \text{ is } \textbf{not} \text{ in the column space of } A.}$