MATH 310, Section 4.6 Solutions

2. $A = [1\ 2\ 3]$

   (a) The rank of A is 1

   (b) A basis for the row space is $\{(1,2,3)\}$

   (c) A basis for the column space is $\{[1]\}$

8. $$A = \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 1 & 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -1 \\ 7 & 14 & -6 & -3 \\ 2 & 4 & -3 & -6 \\ -2 & -4 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

   $$\to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ From this, you can read off all the answers:

   (a) The rank of A is 2.

   (b) Basis for the row space is $\{(1,2,0,3),(0,0,1,4)\}$

   (c) Basis for the column space is $\left\{ \begin{bmatrix} 2 \\ 7 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 1 \\ -1 \end{bmatrix} \right\}$

12. Find a basis for the subspace of $\mathbb{R}^4$ spanned by $S = \{(2,5,-3,-2),(-2,-3,2,-5),(1,3,-2,2),(-1,-5,3,5)\}$.

    For this, we find a basis of the row space of $A = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 2 \\ -2 & -3 & 2 & -5 \\ 2 & 5 & -3 & -2 \\ -1 & -5 & 3 & 5 \end{bmatrix}$

    $$\to \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 6 \\ 0 & 3 & -2 & -1 \\ 0 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -16 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & -1 & 19 \end{bmatrix}$$

    $$\to \begin{bmatrix} 1 & 0 & 1 & -16 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

    From this, you can see that a basis for the row space (hence for the subspace spanned by the rows) is $\boxed{S = \{(1,0,0,3),(0,1,0,-13),(0,0,1,-19)\}}$.

18. Find a basis for and the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}$.

Working this out with Gauss-Jordan, $\begin{bmatrix} 3 & -6 & 21 & 0 \\ -2 & 4 & -14 & 0 \\ 1 & -2 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & 0 \\ -2 & 4 & -14 & 0 \\ 3 & -6 & 21 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, the solutions are $\mathbf{x} = \begin{bmatrix} 2s-7t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$

Thus, $\boxed{\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}\right\} \text{ is a basis for the solution space}}$, and $\boxed{\text{the dimension of the solution space is 2.}}$

19. Determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

The problem is asking if the vector $\mathbf{b}$ is a linear combination of the columns of A.
In other words, it is asking if the following system has a solution:

$$x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solving this in the usual way gives:

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1/2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$$

As you can now see, there are no solutions to this equation, so $\boxed{\mathbf{b} \text{ is } \textbf{not} \text{ in the column space of } A.}$

MATH 310, Section 4.7 Solutions

6. Suppose $B = \{(4,0,7,3), (0,5,-1,-1), (-3,4,2,1), (0,1,5,0)\}$ and $[\mathbf{x}]_B = [-2,3,4,1]^T$.

Since $[\mathbf{x}]_B = [-2,3,4,1]^T = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$, it follows

$\mathbf{x} = -2(4,0,7,3) + 3(0,5,-1,-1) + 4(-3,4,2,1) + 1(0,1,5,0) = (-20,32,-4,-5)$.
Given that the standard basis for $\mathbb{R}^4$ is $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, we have

$$\boxed{[\mathbf{x}]_S = [-20,32,-4,-5]^T = \begin{bmatrix} -20 \\ 32 \\ -4 \\ -5 \end{bmatrix}.}$$

8. Find the coordinate of $\mathbf{x} = (-26, 32)$ relative to the basis $B = \{(-6, 7), (4, -3)\}$.
This involves finding a solution to $x(-6, 7) + y(4, -3) = (-26, 32)$, which gives rise to the system:

$$\begin{cases} -6x + 4y &= -26 \\ 7x - 3y &= 32 \end{cases}$$

$$\begin{bmatrix} -6 & 4 & -26 \\ 7 & -3 & 32 \end{bmatrix} \to \begin{bmatrix} -6 & 4 & -26 \\ 1 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 6 \\ -6 & 4 & -26 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 6 \\ 0 & 10 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

Thus $5(-6, 7) + 1(4, -3) = (-26, 32)$, so $\boxed{[\mathbf{x}]_B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.}$

14. Find the transition matrix from $B = \{(1, 0), (0, 1)\}$ to $B' = \{(1, 1), (5, 6)\}$.
Setting up the problem as in Theorem 4.21, we get

$$\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 1 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 6 & -5 \\ 0 & 1 & -1 & 1 \end{array}\right],$$ so $\boxed{\text{the transition matrix is } P^{-1} = \begin{bmatrix} 6 & -5 \\ -1 & 1 \end{bmatrix}}$

30. Find the coordinate matrix of $p = 3x^2 + 114x + 13$ relative to the standard basis $S = \{1, x, x^2\}$ of P_2.

Since $p = 13 \cdot 1 + 114 \cdot x + 3 \cdot x^2$, it follows that $\boxed{[p]_S = \begin{bmatrix} 13 \\ 114 \\ 3 \end{bmatrix}.}$

36. Suppose P is the transition matrix from B'' to B' and Q is the transition matrix from B' to B. What is the transition matrix from B to B''?

Since P is the transition matrix from B'' to B', we have $P[\mathbf{x}]_{B''} = [\mathbf{x}]_{B'}$, so

$$[\mathbf{x}]_{B''} = P^{-1}[\mathbf{x}]_{B'}. \tag{1}$$

Since Q is the transition matrix from B' to B, we have $Q[\mathbf{x}]_{B'} = [\mathbf{x}]_B$, so

$$[\mathbf{x}]_{B'} = Q^{-1}[\mathbf{x}]_B. \tag{2}$$

Taking equation (1) and replacing the $[\mathbf{x}]_{B'}$ with $Q^{-1}[\mathbf{x}]_B$ (By equation (2)) we get

$$[\mathbf{x}]_{B''} = P^{-1}Q^{-1}[\mathbf{x}]_B. \tag{3}$$

From this, $(P^{-1}Q^{-1})[\mathbf{x}]_B = [\mathbf{x}]_{B''}$, which means $\boxed{P^{-1}Q^{-1} = (QP)^{-1} \text{ is the transition matrix from } B \text{ to } B''.}$