MATH 310, Section 4.5 Solutions

4. A standard basis for P_4 is $\{1, x, x^2, x^3, x^4\}$.

18. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix} \right\}$

This set cannot be a basis because it is not linearly independent: A non-trivial linear combination of the matrices equals the zero matrix.

$$5\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 4\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

38. Is $S = \{(1,4,7), (3,0,1), (2,1,2)\}$ a basis for $\mathbb{R}^3$?

First, to see if the vectors are linearly independent, we must examine the solutions of the equation $c_1(1,4,7) + c_2(3,0,1) + c_3(2,1,2) = (0,0,0)$. Solving this in the usual way,

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -12 & -7 & 0 \\ 0 & -20 & -12 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 7/12 & 0 \\ 0 & -20 & -12 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 7/12 & 0 \\ 0 & 0 & -1/3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 7/12 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

At this point you can see that there is only a trivial solution, so the set is **linearly independent.**

To check if the set spans $\mathbb{R}^3$, let (x, y, z) be an arbitrary vector in $\mathbb{R}^3$ and examine the solutions of $c_1(1,4,7) + c_2(3,0,1) + c_3(2,1,2) = (x, y, z)$. Solving this in the usual way,

$$\begin{bmatrix} 1 & 3 & 2 & x \\ 4 & 0 & 1 & y \\ 7 & 1 & 2 & z \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & x \\ 0 & -12 & -7 & * \\ 0 & -20 & -12 & * \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & * \\ 0 & 1 & 7/12 & * \\ 0 & -20 & -12 & * \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & * \\ 0 & 1 & 7/12 & * \\ 0 & 0 & -1/3 & * \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & * \\ 0 & 1 & 7/12 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Here we haven't bothered to follow through with the values in the fourth column, for we don't need them to answer the question. From the form of the final matrix, you can see that the equation $c_1(1,4,7) + c_2(3,0,1) + c_3(2,1,2) = (x, y, z)$ has a solution, so the set S spans $\mathbb{R}^3$.

Thus $\boxed{\text{the set } S = \{(1,4,7), (3,0,1), (2,1,2)\} \text{ is a basis for } \mathbb{R}^3}$ because it is a linearly independent spanning set.

Finally, we are asked to write $(8,3,8)$ as a linear combination of elements of S. For this we can just redo the work for checking that S is a spanning set, but with $(8,3,8)$ replacing (x, y, z).

$$\begin{bmatrix} 1 & 3 & 2 & 8 \\ 4 & 0 & 1 & 3 \\ 7 & 1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 8 \\ 0 & -12 & -7 & -29 \\ 0 & -20 & -12 & -48 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 8 \\ 0 & 1 & 7/12 & 29/12 \\ 0 & -20 & -12 & -48 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 8 \\ 0 & 1 & 7/12 & 29/12 \\ 0 & 0 & -1/3 & 1/3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 3 & 2 & 8 \\ 0 & 1 & 7/12 & 29/12 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This gives $1(1,4,7) + 3(3,0,1) - (2,1,2) = (8,3,8)$, which checks back.

44. Find a basis for the set of all 3×3 symmetric matrices. What is the dimension of this space?

Recall that a matrix A is symmetric if $A^T = A$.

Thus, a 3×3 symmetric matrix has the form $\begin{bmatrix} u & v & w \\ v & x & y \\ w & y & z \end{bmatrix}$.

This matrix equals the following sum:

$$u \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + v \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + w \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the following set spans the space of symmetric matrices:

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Moreover, we claim that this set is linearly independent. To see this, look at the following equation:

$$u \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + v \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + w \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This results in $\begin{bmatrix} u & v & w \\ v & x & y \\ w & y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which gives only the trivial solution $u = 0$, $v = 0$, $w = 0$, $x = 0$, $y = 0$, $z = 0$. Thus the set S is linearly independent.

Since S is linearly independent and spans the space of symmetric matrices, it follows that S is a basis for the space of symmetric matrices.

Conclusion: The space of 3×3 symmetric matrices is $\boxed{\text{six-dimensional.}}$

52. $W = \{(2s - t, s, t) : s, t \in \mathbb{R}\}$.

(a) Since $(2s - t, s, t) = s(2, 1, 0) + t(-1, 0, 1)$, it follows that $W = \text{Span}\{(2, 1, 0), (-1, 0, 1)\}$.
Notice the vectors $(2, 1, 0)$ and $(-1, 0, 1)$ are linearly independent (because the two vectors are not multiples of one another) so $\boxed{W \text{ is a plane in } \mathbb{R}^3.}$

(b) $\boxed{S = \{(2, 1, 0), (-1, 0, 1)\} \text{ is a basis for } W}$ because these vectors a linearly independent and span W.

(C) $\boxed{\text{The dimension of the subspace is 2}}$ because a basis for it contains two vectors.