MATH 310, Section 4.4 Solutions

16. Does the set $\{(1,0,3),(2,0,-1),(4,0,5),(2,0,6)\}$ span $\mathbb{R}^3$?

    NO. Notice that any linear combination of these four vectors will be a vector whose second component is 0. Since $\mathbb{R}^3$ has vectors whose second component is not 0, then the given set cannot span $\mathbb{R}^3$.

26. Determine if the set $\{(1,0,0),(0,4,0),(0,0,-6),(1,5,-3)\}$ is linearly independent or dependent.

    By inspection we can see $1(1,0,0)+\frac{5}{4}(0,4,0)+\frac{1}{2}(0,0,-6)-1(1,5,-3)=(0,0,0)$.

    Thus the equation $c_1(1,0,0)+c_2(0,4,0)+c_3(0,4,0)-c_4(1,5,-3)=(0,0,0)$ has nontrivial solutions.

    Thus the set $\{(1,0,0),(0,4,0),(0,0,-6),(1,5,-3)\}$ is LINEARLY DEPENDENT.

36. Decide if the matrices $\begin{bmatrix}1&-1\\4&5\end{bmatrix}, \begin{bmatrix}4&3\\-2&3\end{bmatrix}, \begin{bmatrix}1&-8\\22&23\end{bmatrix}$ are inearly independent or dependent.

    We need to see whether the equation $c_1\begin{bmatrix}1&-1\\4&5\end{bmatrix}+c_2\begin{bmatrix}4&3\\-2&3\end{bmatrix}+c_3\begin{bmatrix}1&-8\\22&23\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ has any nontrivial solutions.

    This equation gives rise to the following system.

$$\begin{cases} c_1 + 4c_2 + c_3 = 0 \\ -c_1 + 3c_2 - 8c_3 = 0 \\ 4c_1 - 2c_2 + 22c_3 = 0 \\ 5c_1 + 3c_2 + 23c_3 = 0 \end{cases} \qquad \begin{bmatrix}1&4&1&0\\-1&3&-8&0\\4&-2&22&0\\5&+3&23&0\end{bmatrix} \to \begin{bmatrix}1&4&1&0\\0&7&-7&0\\0&-18&-18&0\\0&-17&-18&0\end{bmatrix} \to$$

$$\begin{bmatrix}1&4&1&0\\0&1&-1&0\\0&1&1&0\\0&-17&-18&0\end{bmatrix} \to \begin{bmatrix}1&4&1&0\\0&1&-1&0\\0&0&2&0\\0&0&-1&0\end{bmatrix} \to \begin{bmatrix}1&4&1&0\\0&1&-1&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

    At this point you can see there's only the trivial solution $c_1=0,\ c_2=0,\ c_3=0$, so the three matrices are LINEARLY INDEPENDENT.

38. Determine if the set $S=\{x^2-2x, x^3+8, x^3-x^2, x^2-4\}$ spans P_3.

    Given an arbitrary polynomial ax^3+bx^2+cx+d, we want to know if we can always find values for c_1,c_2,c_3,c_4 satisfying $c_1(x^2-2x)+c_2(x^3+8)+c_3(x^3-x^2)+c_4(x^2-4)=ax^3+bx^2+cx+d$. Combining, we get $(c_2+c_3)x^3+(c_1-c_3+c_4)x^2-2c_1x+(8c_2-4c_4)=ax^3+bx^2+cx+d$, which leads to the following system.

$$\begin{cases} c_2 + c_3 = a \\ c_1 - c_3 + c_4 = b \\ -2c_1 = c \\ 8c_2 - 4c_4 = d \end{cases} \qquad \begin{bmatrix}0&1&1&0\\1&0&-1&1\\-2&0&0&0\\0&8&0&-4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

    Notice this is of form $A\mathbf{x}=\mathbf{b}$. Expanding along row 3, we get $|A|=-2\begin{vmatrix}1&1&0\\0&-1&1\\8&0&-4\end{vmatrix}=-24\neq 0$.

    Hence the A is invertible, so the system $A\mathbf{x}=\mathbf{b}$ has a solution no matter what $\mathbf{b}$ is. Thus, the conclusion is Yes, the set S spans P_3.

56. Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are three vectors in a vector space. Is the set $S=\{\mathbf{v}-\mathbf{u},\mathbf{w}-\mathbf{v},\mathbf{u}-\mathbf{w}\}$ linearly independent or dependent?

    Notice that $1(\mathbf{v}-\mathbf{u})+1(\mathbf{w}-\mathbf{v})+1(\mathbf{u}-\mathbf{w})=\mathbf{0}$, so it follows the set S is linearly dependent.