MATH 310, Section 4.3 Solutions

4. Show the set W of all 3×2 matrices of form $\begin{bmatrix} a & b \\ a+b & 0 \\ 0 & c \end{bmatrix}$ is a subspace of $V = M_{3,2}$.

   1. Suppose A and B are in W. This means $A = \begin{bmatrix} x & y \\ x+y & 0 \\ 0 & z \end{bmatrix}$ and $B = \begin{bmatrix} x' & y' \\ x'+y' & 0 \\ 0 & z' \end{bmatrix}$ for some real numbers x, y, z, x', y', z'. Then $A+B = \begin{bmatrix} x+x' & y+y' \\ (x+x)'+(y+y') & 0 \\ 0 & z+z' \end{bmatrix}$ has the form of a matrix in W, so $A+B$ is in W.

   2. Suppose $A = \begin{bmatrix} x & y \\ x+y & 0 \\ 0 & z \end{bmatrix}$ is in W and c is a real number. Then $cA = \begin{bmatrix} cx & cy \\ cx+cy & 0 \\ 0 & cz \end{bmatrix}$ has the form of a matrix in W, so cA is in W.

   It follows from Theorem 4.5 that W is a subspace of $M_{3,2}$.

6. Show the set W of differentiable functions on $[0,1]$ is a subspace of all continuous functions on $[0,1]$.

   From Calculus, we know that every differentiable function is also continuous, so W really is a subset of all continuous functions on $[0,1]$. We also know that (1) the sum of two differentiable functions is differentiable, and (2) a scalar multiple of a differentiable function is differentiable. Thus W is closed under addition and scalar multiplication, so W is a subspace by Theorem 4.5.

12. W is the set of all matrices $M_{n,n}$ such that $A^2 = A$.

    Notice that I is in W because $I^2 = I$. But $2I$ is not in W because $(2I)^2 = 4I \neq 2I$. Thus W is not closed under scalar multiplication, so it is not a subspace.

16. Show $W = \{(x_1, x_2, 4) : \ x_1, x_2 \text{ are real numbers}\}$ is NOT a subspace of $\mathbb{R}^3$.

    Notice that vectors $\mathbf{u} = (0,0,4)$ in W, but $2\mathbf{u} = (0,0,8)$ is not in W. Since W is not closed under scalar multiplication, W is not a subspace of $\mathbb{R}^3$.

26. Suppose A is a fixed $m \times n$ matrix. Prove that the set $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

    Proof.
    1. Suppose $\mathbf{u}$ and $\mathbf{v}$ are in W. This means $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Observe that by properties of matrix multiplication, $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$. But $A(\mathbf{u}+\mathbf{v}) = \mathbf{0}$ means $\mathbf{u}+\mathbf{v}$ is in W.

    2. Suppose $\mathbf{u}$ is in W and c is a scalar. The fact that $\mathbf{u}$ is in W means $A\mathbf{u} = \mathbf{0}$. Then $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}$. But $A(c\mathbf{u}) = \mathbf{0}$ means $c\mathbf{u}$ is in W.

    It follows from Theorem 4.5 that W is a subspace of $\mathbb{R}^n$.