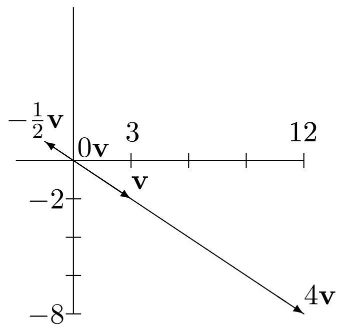
MATH 310, Section 4.1 Solutions

8. 

20. $\mathbf{u} = (0, 4, 3, 4, 4)$ and $\mathbf{v} = (6, 8, -3, 3 - 5)$

    (a) $\mathbf{u} - \mathbf{v} = (-6, -4, 6, 1, 9)$

    (b) $2(\mathbf{u} + 3\mathbf{v}) = 2\mathbf{u} + 6\mathbf{v} = (0, 8, 6, 8, 8) + (36, 48, -18, 18 - 15) = (36, 56, -12, 26, -22)$

26. $\mathbf{w} + \mathbf{u} = -\mathbf{v}$
    $\mathbf{w} = -\mathbf{u} - \mathbf{v} = -(1, -1, 0, 1) - (0, 2, 3, -1) = (-1, -1, -3, 0)$

36. Suppose $\mathbf{u}_1 = (1, 3, 5)$, $\mathbf{u}_2 = (2, -1, 3)$, $\mathbf{u}_3 = (-3, 2, -4)$ and $\mathbf{v} = (-1, 7, 2)$.
    Is $\mathbf{v}$ a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$?
    In other words are there scalars x, y and z with $x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 = \mathbf{v}$?
    This is $x(1, 3, 5) + y(2, -1, 3) + z(-3, 2, -4) = (-1, 7, 2)$ or
    $(x + 2y - 3z,\ \ 3x - y + 2z,\ \ 5x + 3y - 4z) = (-1, 7, 2)$, which is the system

$$\begin{cases} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix} \begin{array}{c} R2 - 3R_1 \to R_2 \\ R3 - 5R_1 \to R_3 \end{array} \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{bmatrix} R_3 - R_2 \to R_3 \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The last row shows there are no solutions. Thus $\mathbf{v}$ is NOT a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

50. Property 3: $0\mathbf{v} = \mathbf{0}$.

Proof. First note that $0\mathbf{v} = 0\mathbf{v}$. From this it follows that

$$\begin{array}{rcll} 0\mathbf{v} &=& (0+0)\mathbf{v} & a.\ \underline{\text{because } 0 + 0 = 0} \\ 0\mathbf{v} &=& 0\mathbf{v} + 0\mathbf{v} & b.\ \underline{\text{distributive property 8}} \\ 0\mathbf{v} + (-0\mathbf{v}) &=& (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v}) & c.\ \underline{\text{add equal quantities to both sides}} \\ \mathbf{0} &=& 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v})) & d.\ \underline{\text{properties 5 and 3}} \\ \mathbf{0} &=& 0\mathbf{v} + \mathbf{0} & e.\ \underline{\text{property 5}} \\ \mathbf{0} &=& 0\mathbf{v} & f.\ \underline{\text{property 4}} \end{array}$$

MATH 310, Section 4.2 Solutions

18. Consider the set $V = \{(x, y) : x \geq 0, y \text{ is a real number}\}$.
In words, V is the right-hand half (1st and 3rd quadrants) of $\mathbb{R}^2$.
This is NOT a vector space because:
Axiom 5 fails: If $\mathbf{u}$ is in V, then $-\mathbf{u}$ will lie outside of V.
Axiom 6 fails: If $\mathbf{u}$ is in V, then $c\mathbf{u}$ may lie outside of V if $c \leq 0$.

20. Consider the set V of all 2×2 matrices of form $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$ (with standard operations).
This is NOT a vector space because:
Axiom 1 fails: Given two vectors $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$ and $\begin{bmatrix} d & e \\ f & 1 \end{bmatrix}$ in V their sum $\begin{bmatrix} a+d & b+e \\ c+f & 2 \end{bmatrix}$ is not in V.
Axiom 4 fails: The zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in V.
Axiom 5 fails: If $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$ is in V, then $-\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -1 \end{bmatrix}$ is not in V.
Axiom 6 fails: If $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$ is in V, then $d\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} = \begin{bmatrix} da & db \\ dc & d \end{bmatrix}$ is not in V when $d \neq 1$.

22. Consider the set V of all 2×2 nonsingular matrices (with standard operations).
This is NOT a vector space because:
Axiom 1 fails: Given vectors $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in V, their sum $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in V.
Axiom 4 fails: The zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in V.
Axiom 6 fails: The scalar multiple $k\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not in V if $k = 0$.

30. Let V be the set of all positive real numbers.
Suppose V has the following operations:
Addition: $x + y = xy$
Scalar multiplication: $cx = x^c$
Is V a vector space?

Observe that all the vector space axioms hold:

1. If $x, y \in V$ then x and y are positive, so xy is positive, so $x + y = xy \in V$.
2. $x + y = xy = yx = y + x$
3. $(x + y) + z = (xy) + z = (xy)z = x(yz) = x + (yz) = x + (y + z)$
4. Notice that $1 \in V$ and the zero vector is $\mathbf{0} = 1$ because $x + \mathbf{0} = x + 1 = x \cdot 1 = x$ for all x in V.
5. If $x \in V$, then $-x = \frac{1}{x}$ because $x + (-x) = x(-x) = x\frac{1}{x} = 1 = \mathbf{0}$
6. If $x \in V$, and $c \in \mathbb{R}$, then $cx = x^c$ is positive because x is positive, so $cx \in V$.
7. $c(x + y) = (x + y)^c = (xy)^c = x^c y^c = x^c + y^c = cx + cy$
8. $(c + d)x = x^{c+d} = x^c x^d = x^c + x^d = cx + dy$
9. Note $c(dx) = c(x^d) = (x^d)^c = x^{cd} = (cd)x$
10. For every $x \in V$, we have $1x = x^1 = x$.

All axioms hold. Thus, the answer is YES, V is a vector space.

34. Property 1: $0\mathbf{v} = \mathbf{0}$.

Proof. First note that $0\mathbf{v} = 0\mathbf{v}$. From this it follows that

$$\begin{aligned}
0\mathbf{v} &= (0+0)\mathbf{v} && \text{(because } 0+0=0\text{)} \\
0\mathbf{v} &= 0\mathbf{v} + 0\mathbf{v} && \text{(distributive property 8)} \\
0\mathbf{v} + (-0\mathbf{v}) &= (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v}) && \text{(add equal quantities to both sides)} \\
\mathbf{0} &= 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v})) && \text{(properties 5 and 3)} \\
\mathbf{0} &= 0\mathbf{v} + \mathbf{0} && \text{(property 5)} \\
\mathbf{0} &= 0\mathbf{v} && \text{(property 4)}
\end{aligned}$$

This proves that $0\mathbf{v} = \mathbf{0}$ in any vector space.