SECTION 1.2 (Let me know if you see any typos and I'll correct them. –RH)

16. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ corresponds to the system $\left\{\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 0 \\ & & & + & z & = & -1 \\ & & & & 0 & = & 0 \end{array}\right.$

The second equation says $z = -1$. Back-substituting this into the first equation gives $x + 2y - 1 = 0$, or $x = 1 - 2y$. Thus the solution is $\boxed{x = 1 - 2t, \quad y = t, \quad z = -1}$

28. Solve the system $\left\{\begin{array}{rcrcrcr} 2x_1 & & & + & 3x_3 & = & 3 \\ 4x_1 & - & 3x_2 & + & 7x_3 & = & 5 \\ 8x_1 & - & 9x_2 & + & 15x_3 & = & 10 \end{array}\right.$

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} \begin{array}{c} R2 - 2R_1 \to R_2 \\ R3 - 4R_1 \to R_3 \end{array} \begin{bmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{bmatrix} \; R3 - 3R_2 \to R_3 \; \begin{bmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Even though this is not in row echelon form, it is simple enough that we can draw a conclusion. The last row gives the equation $0x_1 + 0x_2 + 0x_3 = 1$, or $0 = 1$. Thus the system has **NO SOLUTIONS**.

32. Solve the system $\left\{\begin{array}{rcrcrcrcr} 2x & + & y & - & z & + & 2w & = & -6 \\ 3x & + & 4y & & & + & w & = & 1 \\ x & + & 5y & + & 2z & + & 6w & = & -3 \\ 5x & + & 2y & - & z & - & w & = & 3 \end{array}\right.$

$$\begin{bmatrix} 2 & 1 & -1 & 2 & -6 \\ 3 & 4 & 0 & 1 & 1 \\ 1 & 5 & 2 & 6 & -3 \\ 5 & 2 & -1 & -1 & 3 \end{bmatrix} \begin{array}{c} R_1 \leftrightarrow R_3 \\ \text{(gets 1 in} \\ \text{upper-left)} \end{array} \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 3 & 4 & 0 & 1 & 1 \\ 2 & 1 & -1 & 2 & -6 \\ 5 & 2 & -1 & -1 & 3 \end{bmatrix} \begin{array}{c} R_2 - 3R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \\ R_4 - 5R_1 \to R_4 \end{array}$$

$$\begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & -11 & -6 & -17 & 10 \\ 0 & -9 & -5 & -10 & 0 \\ 0 & -23 & -11 & -31 & 18 \end{bmatrix} \begin{array}{c} -R_2 \to R_2 \\ -R_3 \to R_3 \\ -R_4 \to R_4 \\ \text{(gets rid of} \\ \text{negatives)} \end{array} \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 11 & 6 & 17 & -10 \\ 0 & 9 & 5 & 10 & 0 \\ 0 & 23 & 11 & 31 & -18 \end{bmatrix} \begin{array}{c} R_4 - 2R_2 \to R_4 \\ \text{(gets a 1 in } 2^{nd} \\ \text{column)} \end{array}$$

$$\begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 11 & 6 & 17 & -10 \\ 0 & 9 & 5 & 10 & 0 \\ 0 & 1 & -1 & -3 & 2 \end{bmatrix} \; R_2 \leftrightarrow R_4 \; \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & -1 & -3 & 2 \\ 0 & 9 & 5 & 10 & 0 \\ 0 & 11 & 6 & 17 & -10 \end{bmatrix} \begin{array}{c} R_3 - 9R_2 \to R_3 \\ R_4 - 11R_2 \to R_4 \end{array}$$

$$\begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & -1 & -3 & 2 \\ 0 & 0 & -14 & -37 & 18 \\ 0 & 0 & -17 & -50 & 32 \end{bmatrix} \; -\tfrac{1}{14}R_3 \to R_3 \; \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & -1 & -3 & 2 \\ 0 & 0 & 1 & \frac{37}{14} & -\frac{18}{14} \\ 0 & 0 & -17 & -50 & 32 \end{bmatrix} \; R_4 + 17R_3 \to R_4$$

$$\begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & -1 & -3 & 2 \\ 0 & 0 & 1 & \frac{37}{14} & -\frac{18}{14} \\ 0 & 0 & 0 & -\frac{71}{14} & \frac{142}{14} \end{bmatrix} \; -\tfrac{14}{71}R_4 \to R_4 \; \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & -1 & -3 & 2 \\ 0 & 0 & 1 & \frac{37}{14} & -\frac{18}{14} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Translating back into a system: $\left\{\begin{array}{rcrcrcrcr} x & + & 5y & + & 2z & + & 6w & = & -3 \\ & & y & - & z & - & 3w & = & 2 \\ & & & & z & + & \frac{37}{14}w & = & -\frac{18}{14} \\ & & & & & & w & = & -2 \end{array}\right.$

Back-substitution gives the solution: $\boxed{x = 1, y = 0, z = 4, w = -2}$

40. The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ is a *coefficient* matrix for a *homogeneous* system. Thus the fourth column consists of coefficients of the fourth variable; the constant terms of the two equations are understood to all be zeros and do not appear as a last column. Thus this matrix (which is already in reduced row echelon form) corresponds to the system $\begin{cases} x + 0y + 0z + 0w = 0 \\ 0x + y + z + 0w = 0 \end{cases}$

The solutions can be read off as $\boxed{x = 0 \ , y = -t \ , z = t \ , w = s.}$

44. The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ corresponds to the system system $\begin{cases} 0x + 0y = 0 \\ 0x + 0y = 0 \\ 0x + 0y = 0 \end{cases}$

Any values of x and y automatically satisfy this system,
so the solutions are $x = s, \quad y = t$ for any real numbers s and t.

62. Solve the system $\begin{cases} x^2 + 2y^3 = 2 \\ 3x^2 - y^3 = 13 \end{cases}$

This system is not linear, so we begin by making the substitutions $u = x^2$ and $v = y^3$.
The new system is $\begin{cases} u + 2v = 2 \\ 3u - v = 13 \end{cases}$ It is linear and it is solved as follows.

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

Thus the solution to the new system is $u = 4$, $v = -1$. This means $x^2 = 4$ and $y^3 = -1$, so $x = \pm 2$ and $y = -1$. Therefore, the original system has TWO solutions $\boxed{x = 2, y = -1 \text{ and } x = -2, y = -1.}$

Editorial comment: A linear system would never have just two solutions. A linear system always has zero, one or infinitely many solutions. However, the system we solved here was not a linear system, so it should not be surprising that there were two solutions. –RH