Seventh Edition

1 Systems of Linear Equations

Electrical Network Analysis (p. 30)

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1.1 Introduction to Systems of Linear Equations

Seventh Edition

- Recognize a linear equation in n variables.
- Find a parametric representation of a solution set.
- Determine whether a system of linear equations is consistent or inconsistent.
- Use back-substitution and Gaussian elimination to solve a system of linear equations.

LINEAR EQUATIONS IN n VARIABLES

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally, you will find examples and exercises requiring a knowledge of calculus; these are clearly marked in the text.

Early in your study of linear algebra, you will discover that many of the solution methods involve multiple arithmetic steps, so it is essential to check your work. Use a computer or calculator to check your work and perform routine computations.

Although you will be familiar with some material in this chapter, you should carefully study the methods presented in this chapter. This will cultivate and clarify your intuition for the more abstract material that follows.

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \quad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** x and y. Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \quad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

This is a **linear equation in three variables** x, y, and z. In general, a linear equation in n variables is defined as follows.

REMARK

Letters that occur early in the alphabet are used to represent constants, and letters that occur late in the alphabet are used to represent variables.

Definition of a Linear Equation in n Variables

A **linear equation in n variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient**, and x_1 is the **leading variable.**

Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

EXAMPLE 1 Linear and Nonlinear Equations

Each equation is linear.

a. $3x + 2y = 7$ **b.** $\frac{1}{2}x + y - \pi z = \sqrt{2}$ **c.** $(\sin \pi)x_1 - 4x_2 = e^2$

Each equation is not linear.

a. $xy + z = 2$ **b.** $e^x - 2y = 4$ **c.** $\sin x_1 + 2x_2 - 3x_3 = 0$

SOLUTIONS AND SOLUTION SETS

A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged to satisfy the equation when you substitute the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \ldots, \quad x_n = s_n$$

into the equation. For example, $x_1 = 2$ and $x_2 = 1$ satisfy the equation $x_1 + 2x_2 = 4$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is called its **solution set,** and when you have found this set, you have **solved** the equation. To describe the entire solution set of a linear equation, use a **parametric representation,** as illustrated in Examples 2 and 3.

EXAMPLE 2 Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

SOLUTION

To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. Solving for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable x_2 is **free,** which means that it can take on any real value. The variable x_1 is not free because its value depends on the value assigned to x_2. To represent the infinitely many solutions of this equation, it is convenient to introduce a third variable t called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \quad x_2 = t, \quad t \text{ is any real number.}$$

To obtain particular solutions, assign values to the parameter t. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$. ■

To parametrically represent the solution set of the linear equation in Example 2 another way, you could have chosen x_1 to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \quad x_2 = 2 - \tfrac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, choose the variables that occur last in a given equation to be free variables.

EXAMPLE 3 Parametric Representation of a Solution Set

Solve the linear equation $3x + 2y - z = 3$.

SOLUTION

Choosing y and z to be the free variables, solve for x to obtain

$$\begin{aligned} 3x &= 3 - 2y + z \\ x &= 1 - \tfrac{2}{3}y + \tfrac{1}{3}z. \end{aligned}$$

Letting $y = s$ and $z = t$, you obtain the parametric representation

$$x = 1 - \tfrac{2}{3}s + \tfrac{1}{3}t, \quad y = s, \quad z = t$$

where s and t are any real numbers. Two particular solutions are

$$x = 1, y = 0, z = 0 \quad \text{and} \quad x = 1, y = 1, z = 2.$$ ■

SYSTEMS OF LINEAR EQUATIONS

REMARK

The double-subscript notation indicates a_{ij} is the coefficient of x_j in the ith equation.

A **system of *m* linear equations in *n* variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the system. For example, the system

$$\begin{aligned}
3x_1 + 2x_2 &= 3\\
-x_1 + x_2 &= 4
\end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy *both* equations. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

DISCOVERY

1. Graph the two lines

$$\begin{aligned}
3x - y &= 1\\
2x - y &= 0
\end{aligned}$$

in the xy-plane. Where do they intersect? How many solutions does this system of linear equations have?

2. Repeat this analysis for the pairs of lines

$$\begin{aligned}
3x - y &= 1\\
3x - y &= 0
\end{aligned}$$

and

$$\begin{aligned}
3x - y &= 1\\
6x - 2y &= 2.
\end{aligned}$$

3. What basic types of solution sets are possible for a system of two equations in two unknowns?

LINEAR ALGEBRA APPLIED

In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas (CH_4) combines with oxygen (O_2) and burns, carbon dioxide (CO_2) and water (H_2O) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O.$$

Because a chemical reaction can neither create nor destroy atoms, all of the atoms represented on the left side of the arrow must be accounted for on the right side of the arrow. This is called *balancing* the chemical equation. In the given example, chemists can use a system of linear equations to find values of x_1, x_2, x_3, and x_4 that will balance the chemical equation.

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It is possible for a system of linear equations to have exactly one solution, infinitely many solutions, or no solution. A system of linear equations is **consistent** when it has at least one solution and **inconsistent** when it has no solution.

EXAMPLE 4 Systems of Two Equations in Two Variables

Solve and graph each system of linear equations.

a. $x + y = 3$, $x - y = -1$ **b.** $x + y = 3$, $2x + 2y = 6$ **c.** $x + y = 3$, $x + y = 1$

$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned} \qquad \begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned} \qquad \begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

SOLUTION

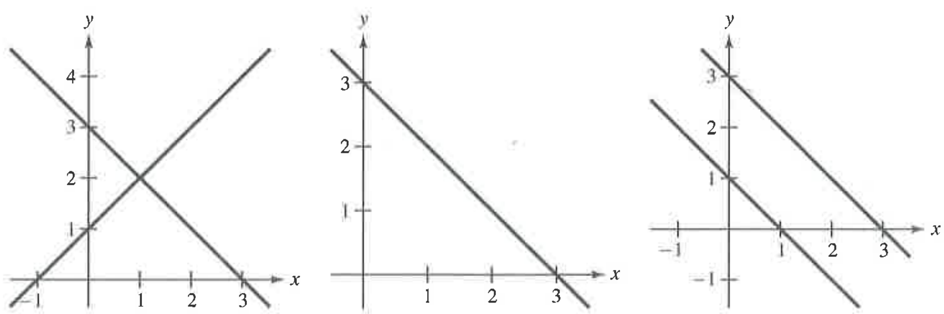
a. This system has exactly one solution, $x = 1$ and $y = 2$. One way to obtain the solution is to add the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is two *intersecting* lines, as shown in Figure 1.1(a).

b. This system has infinitely many solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is

$$x = 3 - t, \quad y = t, \quad t \text{ is any real number.}$$

The graph of this system is two *coincident* lines, as shown in Figure 1.1(b).

c. This system has no solution because the sum of two numbers cannot be 3 and 1 simultaneously. The graph of this system is two *parallel* lines, as shown in Figure 1.1(c).



a. Two intersecting lines: $x + y = 3$, $x - y = -1$

b. Two coincident lines: $x + y = 3$, $2x + 2y = 6$

c. Two parallel lines: $x + y = 3$, $x + y = 1$

Figure 1.1

Example 4 illustrates the three basic types of solution sets that are possible for a system of linear equations. This result is stated here without proof. (The proof is provided later in Theorem 2.5.)

Number of Solutions of a System of Linear Equations

For a system of linear equations, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has infinitely many solutions (consistent system).
3. The system has no solution (inconsistent system).

SOLVING A SYSTEM OF LINEAR EQUATIONS

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad\quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad\qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

EXAMPLE 5 Using Back-Substitution in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{aligned} x - 2y &= 5 \qquad &\text{Equation 1} \\ y &= -2 \qquad &\text{Equation 2} \end{aligned}$$

SOLUTION

From Equation 2, you know that $y = -2$. By substituting this value of y into Equation 1, you obtain

$$\begin{aligned} x - 2(-2) &= 5 \qquad &\text{Substitute } -2 \text{ for } y. \\ x &= 1. \qquad &\text{Solve for } x. \end{aligned}$$

The system has exactly one solution: $x = 1$ and $y = -2$.

The term *back-substitution* implies that you work *backwards*. For instance, in Example 5, the second equation gives you the value of y. Then you substitute that value into the first equation to solve for x. Example 6 further demonstrates this procedure.

EXAMPLE 6 Using Back-Substitution in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \qquad &\text{Equation 1} \\ y + 3z &= 5 \qquad &\text{Equation 2} \\ z &= 2 \qquad &\text{Equation 3} \end{aligned}$$

SOLUTION

From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

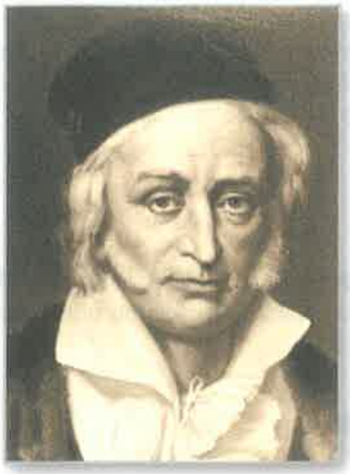
$$\begin{aligned} y + 3(2) &= 5 \qquad &\text{Substitute 2 for } z. \\ y &= -1. \qquad &\text{Solve for } y. \end{aligned}$$

Then, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \qquad &\text{Substitute } -1 \text{ for } y \text{ and 2 for } z. \\ x &= 1. \qquad &\text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

Two systems of linear equations are **equivalent** when they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form by using the operations listed on the next page.



Carl Friedrich Gauss (1777–1855)
German mathematician Carl Friedrich Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. Gauss used a form of what is now known as Gaussian elimination in his research. Although this method was named in his honor, the Chinese used an almost identical method some 2000 years prior to Gauss.

Operations That Produce Equivalent Systems

Each of the following operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 7 Using Elimination to Rewrite a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION

Although there are several ways to begin, you want to use a systematic procedure that is easily applicable to large systems. Work from the upper left corner of the system, saving the x at the upper left and eliminating the other x-terms from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that you have eliminated all but the first x from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \quad y = -1, \quad z = 2.$$

Each of the three equations in Example 7 represents a plane in a three-dimensional coordinate system. Because the unique solution of the system is the point

$$(x, y, z) = (1, -1, 2)$$

the three planes intersect at this point, as shown in Figure 1.2.

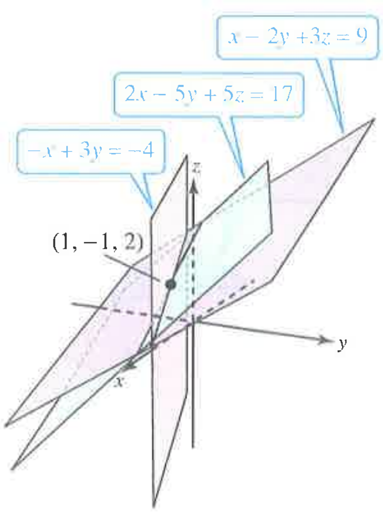


Figure 1.2

Because many steps are required to solve a system of linear equations, it is very easy to make arithmetic errors. So, you should develop the habit of *checking your solution by substituting it into each equation in the original system.* For instance, in Example 7, you can check the solution $x = 1$, $y = -1$, and $z = 2$ as follows.

$$\begin{aligned} &\text{Equation 1:} & (1) - 2(-1) + 3(2) &= 9 \\ &\text{Equation 2:} & -(1) + 3(-1) &= -4 \\ &\text{Equation 3:} & 2(1) - 5(-1) + 5(2) &= 17 \end{aligned}$$

Substitute solution in each equation of the original system.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage of the elimination process, you obtain a false statement such as $0 = -2$.

EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 2 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

SOLUTION

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 5x_2 - 4x_3 &= -2 \end{aligned}$$

Adding -1 times the first equation to the third equation produces a new third equation.

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.)

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, the original system also has no solution.

As in Example 7, the three equations in Example 8 represent planes in a three-dimensional coordinate system. In this example, however, the system is inconsistent. So, the planes do not have a point in common, as shown in Figure 1.3.

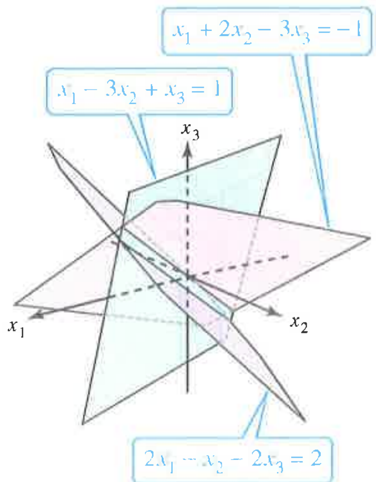


Figure 1.3

This section ends with an example of a system of linear equations that has infinitely many solutions. You can represent the solution set for such a system in parametric form, as you did in Examples 2 and 3.

EXAMPLE 9 **A System with Infinitely Many Solutions**

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 \qquad - 3x_3 &= -1 \\ -x_1 + 3x_2 \qquad &= 1 \end{aligned}$$

SOLUTION

Begin by rewriting the system in row-echelon form, as follows.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 \qquad &= 1 \end{aligned}$$

Interchange the first two equations.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Adding -3 times the second equation to the third equation eliminates the third equation.

Because the third equation is unnecessary, omit it to obtain the system shown below.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \quad x_2 = t, \quad x_3 = t, \quad t \text{ is any real number.}$$

DISCOVERY

1. Graph the two lines represented by the system of equations.

$$\begin{aligned} x - 2y &= 1 \\ -2x + 3y &= -3 \end{aligned}$$

2. Use Gaussian elimination to solve this system as follows.

$$\begin{aligned} x - 2y &= 1 \\ -1y &= -1 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

Graph the system of equations you obtain at each step of this process. What do you observe about the lines?

You are asked to repeat this graphical analysis for other systems in Exercises 89 and 90.

1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Linear Equations In Exercises 1–6, determine whether the equation is linear in the variables x and y.

1. $2x - 3y = 4$
2. $3x - 4xy = 0$
3. $\frac{3}{y} + \frac{2}{x} - 1 = 0$
4. $x^2 + y^2 = 4$
5. $2 \sin x - y = 14$
6. $(\sin 2)x - y = 14$

Parametric Representation In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

7. $2x - 4y = 0$
8. $3x - \frac{1}{2}y = 9$
9. $x + y + z = 1$
10. $13x_1 - 26x_2 + 39x_3 = 13$

Graphical Analysis In Exercises 11–24, graph the system of linear equations. Solve the system and interpret your answer.

11. $\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$
12. $\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$
13. $\begin{aligned} x - y &= 1 \\ -2x + 2y &= 5 \end{aligned}$
14. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$
15. $\begin{aligned} 3x - 5y &= 7 \\ 2x + y &= 9 \end{aligned}$
16. $\begin{aligned} -x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$
17. $\begin{aligned} 2x - y &= 5 \\ 5x - y &= 11 \end{aligned}$
18. $\begin{aligned} x - 5y &= 21 \\ 6x + 5y &= 21 \end{aligned}$
19. $\begin{aligned} \frac{x+3}{4} + \frac{y-1}{3} &= 1 \\ 2x - y &= 12 \end{aligned}$
20. $\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \\ x - 2y &= 5 \end{aligned}$
21. $\begin{aligned} 0.05x - 0.03y &= 0.07 \\ 0.07x + 0.02y &= 0.16 \end{aligned}$
22. $\begin{aligned} 0.2x - 0.5y &= -27.8 \\ 0.3x + 0.4y &= 68.7 \end{aligned}$
23. $\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$
24. $\begin{aligned} \frac{2x}{3} + \frac{y}{6} &= \frac{2}{3} \\ 4x + y &= 4 \end{aligned}$

Back-Substitution In Exercises 25–30, use back-substitution to solve the system.

25. $\begin{aligned} x_1 - x_2 &= 2 \\ x_2 &= 3 \end{aligned}$
26. $\begin{aligned} 2x_1 - 4x_2 &= 6 \\ 3x_2 &= 9 \end{aligned}$
27. $\begin{aligned} -x + y - z &= 0 \\ 2y + z &= 3 \\ \tfrac{1}{2}z &= 0 \end{aligned}$
28. $\begin{aligned} x - y &= 4 \\ 2y + z &= 6 \\ 3z &= 6 \end{aligned}$
29. $\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$
30. $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$

Graphical Analysis In Exercises 31–36, complete the following for the system of equations.

(a) Use a graphing utility to graph the system.

(b) Use the graph to determine whether the system is consistent or inconsistent.

(c) If the system is consistent, approximate the solution.

(d) Solve the system algebraically.

(e) Compare the solution in part (d) with the approximation in part (c). What can you conclude?

31. $\begin{aligned} -3x - y &= 3 \\ 6x + 2y &= 1 \end{aligned}$
32. $\begin{aligned} 4x - 5y &= 3 \\ -8x + 10y &= 14 \end{aligned}$
33. $\begin{aligned} 2x - 8y &= 3 \\ \tfrac{1}{2}x + y &= 0 \end{aligned}$
34. $\begin{aligned} 9x - 4y &= 5 \\ \tfrac{1}{2}x + \tfrac{1}{3}y &= 0 \end{aligned}$
35. $\begin{aligned} 4x - 8y &= 9 \\ 0.8x - 1.6y &= 1.8 \end{aligned}$
36. $\begin{aligned} -5.3x + 2.1y &= 1.25 \\ 15.9x - 6.3y &= -3.75 \end{aligned}$

System of Linear Equations In Exercises 37–56, solve the system of linear equations.

37. $\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$
38. $\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$
39. $\begin{aligned} 2u + v &= 120 \\ u + 2v &= 120 \end{aligned}$
40. $\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$
41. $\begin{aligned} 9x - 3y &= -1 \\ \tfrac{1}{5}x + \tfrac{2}{5}y &= -\tfrac{1}{3} \end{aligned}$
42. $\begin{aligned} \tfrac{2}{3}x_1 + \tfrac{1}{6}x_2 &= 0 \\ 4x_1 + x_2 &= 0 \end{aligned}$
43. $\begin{aligned} \frac{x-2}{4} + \frac{y-1}{3} &= 2 \\ x - 3y &= 20 \end{aligned}$
44. $\begin{aligned} \frac{x_1+4}{3} + \frac{x_2+1}{2} &= 1 \\ 3x_1 - x_2 &= -2 \end{aligned}$
45. $\begin{aligned} 0.02x_1 - 0.05x_2 &= -0.19 \\ 0.03x_1 + 0.04x_2 &= 0.52 \end{aligned}$
46. $\begin{aligned} 0.05x_1 - 0.03x_2 &= 0.21 \\ 0.07x_1 + 0.02x_2 &= 0.17 \end{aligned}$
47. $\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x - z &= 0 \end{aligned}$
48. $\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$
49. $\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - 3x_2 + 6x_3 &= 8 \end{aligned}$
50. $\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$

The symbol indicates an exercise in which you are instructed to use a graphing utility or a symbolic computer software program.

51. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

52. $$\begin{aligned} x_1 + 4x_3 &= 13 \\ 4x_1 - 2x_2 + x_3 &= 7 \\ 2x_1 - 2x_2 - 7x_3 &= -19 \end{aligned}$$

53. $$\begin{aligned} x - 3y + 2z &= 18 \\ 5x - 15y + 10z &= 18 \end{aligned}$$

54. $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

55. $$\begin{aligned} x + y + z + w &= 6 \\ 2x + 3y - w &= 0 \\ -3x + 4y + z + 2w &= 4 \\ x + 2y - z + w &= 0 \end{aligned}$$

56. $$\begin{aligned} x_1 + 3x_4 &= 4 \\ 2x_2 - x_3 - x_4 &= 0 \\ 3x_2 - 2x_4 &= 1 \\ 2x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

System of Linear Equations **In Exercises 57–60, use a software program or a graphing utility to solve the system of linear equations.**

57. $$\begin{aligned} x_1 + 0.5x_2 + 0.33x_3 + 0.25x_4 &= 1.1 \\ 0.5x_1 + 0.33x_2 + 0.25x_3 + 0.21x_4 &= 1.2 \\ 0.33x_1 + 0.25x_2 + 0.2x_3 + 0.17x_4 &= 1.3 \\ 0.25x_1 + 0.2x_2 + 0.17x_3 + 0.14x_4 &= 1.4 \end{aligned}$$

58. $$\begin{aligned} 120.2x + 62.4y - 36.5z &= 258.64 \\ 56.8x - 42.8y + 27.3z &= -71.44 \\ 88.1x + 72.5y - 28.5z &= 225.88 \end{aligned}$$

59. $$\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{3}{7}x_2 + \tfrac{2}{9}x_3 &= \tfrac{349}{630} \\ \tfrac{2}{3}x_1 + \tfrac{4}{9}x_2 - \tfrac{2}{5}x_3 &= -\tfrac{19}{45} \\ \tfrac{4}{5}x_1 - \tfrac{1}{8}x_2 + \tfrac{4}{3}x_3 &= \tfrac{139}{150} \end{aligned}$$

60. $$\begin{aligned} \tfrac{1}{8}x - \tfrac{1}{7}y + \tfrac{1}{6}z - \tfrac{1}{5}w &= 1 \\ \tfrac{1}{7}x + \tfrac{1}{6}y - \tfrac{1}{5}z + \tfrac{1}{4}w &= 1 \\ \tfrac{1}{6}x - \tfrac{1}{5}y + \tfrac{1}{4}z - \tfrac{1}{3}w &= 1 \\ \tfrac{1}{5}x + \tfrac{1}{4}y - \tfrac{1}{3}z + \tfrac{1}{2}w &= 1 \end{aligned}$$

Number of Solutions **In Exercises 61–64, state why the system of equations must have at least one solution. Then solve the system and determine whether it has exactly one solution or infinitely many solutions.**

61. $$\begin{aligned} 4x + 3y + 17z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

62. $$\begin{aligned} 2x + 3y &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0 \end{aligned}$$

63. $$\begin{aligned} 5x + 5y - z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

64. $$\begin{aligned} 12x + 5y + z &= 0 \\ 12x + 4y - z &= 0 \end{aligned}$$

65. **Nutrition** One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 177.4 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 436.7 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

66. **Airplane Speed** Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane if 2 hours after the first plane departs, the planes are 3200 kilometers apart.

True or False? **In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

67. (a) A system of one linear equation in two variables is always consistent.

(b) A system of two linear equations in three variables is always consistent.

(c) If a linear system is consistent, then it has infinitely many solutions.

68. (a) A linear system can have exactly two solutions.

(b) Two systems of linear equations are equivalent when they have the same solution set.

(c) A system of three linear equations in two variables is always inconsistent.

69. Find a system of two equations in two variables, x_1 and x_2, that has the solution set given by the parametric representation $x_1 = t$ and $x_2 = 3t - 4$, where t is any real number. Then show that the solutions to the system can also be written as

$$x_1 = \frac{4}{3} + \frac{t}{3} \quad \text{and} \quad x_2 = t.$$

70. Find a system of two equations in three variables, x_1, x_2, and x_3, that has the solution set given by the parametric representation

$$x_1 = t, \quad x_2 = s, \quad \text{and} \quad x_3 = 3 + s - t$$

where s and t are any real numbers. Then show that the solutions to the system can also be written as

$$x_1 = 3 + s - t, \quad x_2 = s, \quad \text{and} \quad x_3 = t.$$

The symbol indicates that electronic data sets for these exercises are available at *www.cengagebrain.com*. These data sets are compatible with each of the following technologies: MATLAB, *Mathematica*, *Maple*, TI-83 Plus, TI-84 Plus, TI-89, Voyage 200.

Substitution **In Exercises 71–74, solve the system of equations by letting $A = 1/x$, $B = 1/y$, and $C = 1/z$.**

71. $\frac{12}{x} - \frac{12}{y} = 7$
$\frac{3}{x} + \frac{4}{y} = 0$

72. $\frac{2}{x} + \frac{3}{y} = 0$
$\frac{3}{x} - \frac{4}{y} = -\frac{25}{6}$

73. $\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 4$
$\frac{4}{x} + \frac{2}{z} = 10$
$-\frac{2}{x} + \frac{3}{y} - \frac{13}{z} = -8$

74. $\frac{2}{x} + \frac{1}{y} - \frac{2}{z} = 5$
$\frac{3}{x} - \frac{4}{y} = -1$
$\frac{2}{x} + \frac{1}{y} + \frac{3}{z} = 0$

Trigonometric Coefficients **In Exercises 75 and 76, solve the system of linear equations for x and y.**

75. $(\cos\theta)x + (\sin\theta)y = 1$
$(-\sin\theta)x + (\cos\theta)y = 0$

76. $(\cos\theta)x + (\sin\theta)y = 1$
$(-\sin\theta)x + (\cos\theta)y = 1$

Coefficient Design **In Exercises 77–82, determine the value(s) of k such that the system of linear equations has the indicated number of solutions.**

77. Infinitely many solutions
$4x + ky = 6$
$kx + y = -3$

78. Infinitely many solutions
$kx + y = 4$
$2x - 3y = -12$

79. Exactly one solution
$x + ky = 0$
$kx + y = 0$

80. No solution
$x + ky = 2$
$kx + y = 4$

81. No solution
$x + 2y + kz = 6$
$3x + 6y + 8z = 4$

82. Exactly one solution
$kx + 2ky + 3kz = 4k$
$x + y + z = 0$
$2x - y + z = 1$

83. Determine the values of k such that the system of linear equations does not have a unique solution.
$x + y + kz = 3$
$x + ky + z = 2$
$kx + y + z = 1$

84. CAPSTONE Find values of a, b, and c such that the system of linear equations has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution. Explain your reasoning.
$x + 5y + z = 0$
$x + 6y - z = 0$
$2x + ay + bz = c$

85. **Writing** Consider the system of linear equations in x and y.
$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$
$a_3x + b_3y = c_3$
Describe the graphs of these three equations in the xy-plane when the system has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution.

86. **Writing** Explain why the system of linear equations in Exercise 85 must be consistent when the constant terms c_1, c_2, and c_3 are all zero.

87. Show that if $ax^2 + bx + c = 0$ for all x, then $a = b = c = 0$.

88. Consider the system of linear equations in x and y.
$ax + by = e$
$cx + dy = f$
Under what conditions will the system have exactly one solution?

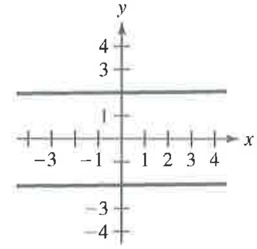
Discovery **In Exercises 89 and 90, sketch the lines represented by the system of equations. Then use Gaussian elimination to solve the system. At each step of the elimination process, sketch the corresponding lines. What do you observe about the lines?**

89. $x - 4y = -3$
$5x - 6y = 13$

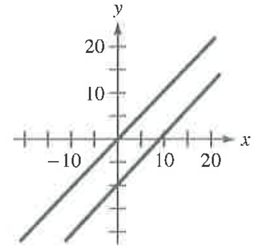
90. $2x - 3y = 7$
$-4x + 6y = -14$

Writing **In Exercises 91 and 92, the graphs of the two equations appear to be parallel. Solve the system of equations algebraically. Explain why the graphs are misleading.**

91. $100y - x = 200$
$99y - x = -198$



92. $21x - 20y = 0$
$13x - 12y = 120$



1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Determine the size of a matrix and write an augmented or coefficient matrix from a system of linear equations.
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations.
- Solve a homogeneous system of linear equations.

MATRICES

Section 1.1 introduced Gaussian elimination as a procedure for solving a system of linear equations. In this section, you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a **matrix.**

REMARK

The plural of matrix is *matrices.* If each entry of a matrix is a *real* number, then the matrix is called a **real matrix.** Unless stated otherwise, assume all matrices in this text are real matrices.

Definition of a Matrix

If m and n are positive integers, an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{c} \text{Row } 1 \\ \text{Row } 2 \\ \text{Row } 3 \\ \vdots \\ \text{Row } m \end{array} \overset{\begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}$$

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry a_{ij} is located in the ith row and the jth column. The index i is called the **row subscript** because it identifies the row in which the entry lies, and the index j is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with m rows and n columns is said to be of **size** $m \times n$. When $m = n$, the matrix is called **square** of **order** n and the entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the **main diagonal** entries.

EXAMPLE 1 **Sizes of Matrices**

Each matrix has the indicated size.

a. Size: 1×1 $\quad [2]$ **b.** Size: 2×2 $\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **c.** Size: 2×3 $\quad \begin{bmatrix} e & 2 & -7 \\ \pi & \sqrt{2} & 4 \end{bmatrix}$

REMARK

Begin by aligning the variables in the equations vertically. Use 0 to indicate coefficients of zero in the matrix. Note the fourth column of constant terms in the augmented matrix.

One common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is called the **augmented matrix** of the system. The matrix containing only the coefficients of the system is called the **coefficient matrix** of the system. Here is an example.

System	Augmented Matrix	Coefficient Matrix
$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x - 4z &= 6 \end{aligned}$	$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$

ELEMENTARY ROW OPERATIONS

In the previous section, you studied three operations that produce equivalent systems of linear equations.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a finite sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operations performed in each step so that checking your work is easier.

Because solving some systems involves several steps, it is helpful to use a shorthand method of notation to keep track of each elementary row operation you perform. The next example introduces this notation.

TECHNOLOGY

Many graphing utilities and software programs can perform elementary row operations on matrices. If you use a graphing utility, you may see something similar to the following for Example 2(c). The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 2(c).

```
A
        [[1 2 -4 3 ]
         [0 3 -2 -1]
         [2 1 5  -2]]

mRAdd(-2,A,1,3)
        [[1 2  -4 3 ]
         [0 3  -2 -1]
         [0 -3 13 -8]]
```

EXAMPLE 2 **Elementary Row Operations**

a. Interchange the first and second rows.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$

b. Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\left(\frac{1}{2}\right)R_1 \rightarrow R_1$

c. Add -2 times the first row to the third row to produce a new third row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$

Notice that adding -2 times row 1 to row 3 does not change row 1.

In Example 7 in Section 1.1, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

EXAMPLE 3 **Using Elementary Row Operations to Solve a System**

Linear System

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Add the first row to the second row to produce a new second row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad R_2 + R_1 \rightarrow R_2$$

Add -2 times the first equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Add -2 times the first row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \quad R_3 + (-2)R_1 \rightarrow R_3$$

Add the second equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Add the second row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad R_3 + R_2 \rightarrow R_3$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Use back-substitution to find the solution, as in Example 6 in Section 1.1. The solution is $x = 1$, $y = -1$, and $z = 2$.

The last matrix in Example 3 is said to be in **row-echelon** form. The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the following properties.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

EXAMPLE 4 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

TECHNOLOGY

Use a graphing utility or a software program to find the reduced row-echelon form of the matrix in Example 4(b). The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 4(b). Similar exercises and projects are also available on the website.

SOLUTION

The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because a row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4(e), multiplying the second row in the matrix by $\frac{1}{2}$ changes the matrix to row-echelon form.

The following summarizes the procedure for using Gaussian elimination with back-substitution.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which you perform the elementary row operations is important. Operate from *left to right by columns*, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

LINEAR ALGEBRA APPLIED

The Global Positioning System (GPS) is a network of 24 satellites originally developed by the U.S. military as a navigational tool. Today, GPS receivers are used in a wide variety of civilian applications, such as determining directions, locating vessels lost at sea, and monitoring earthquakes. A GPS receiver works by using satellite readings to calculate its location. In three dimensions, the receiver uses signals from at least four satellites to "trilaterate" its position. In a simplified mathematical model, a system of three linear equations in four unknowns (three dimensions and time) is used to determine the coordinates of the receiver as functions of time.

EXAMPLE 5 Gaussian Elimination with Back-Substitution

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \qquad &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Interchange the first two rows. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Adding -2 times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

Adding -1 times the first row to the fourth row produces a new fourth row. $R_4 + (-1)R_1 \rightarrow R_4$

Now that the first column is in the desired form, change the second column as follows.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Adding 6 times the second row to the fourth row produces a new fourth row. $R_4 + (6)R_2 \rightarrow R_4$

To write the third and fourth columns in proper form, multiply the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiplying the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$ produces new third and fourth rows. $(\frac{1}{3})R_3 \rightarrow R_3$ $(-\frac{1}{13})R_4 \rightarrow R_4$

The matrix is now in row-echelon form, and the corresponding system is as follows.

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Use back-substitution to find that the solution is $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, and $x_4 = 3$.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then it is unnecessary to continue the elimination process. You can simply conclude that the system has no solution, or is *inconsistent.*

EXAMPLE 6 A System with No Solution

Solve the system.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_4 + (-3)R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is *inconsistent.* You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third equation is not possible, the system has no solution.

GAUSS-JORDAN ELIMINATION

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination,** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. Example 7 demonstrates this procedure.

EXAMPLE 7 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION

In Example 3, you used Gaussian elimination to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as follows.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_2 + (-3)R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (-9)R_3 \rightarrow R_1$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2. \end{aligned}$$

The elimination procedures described in this section sometimes result in fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ 3x - 2y + 3z &= 11 \\ -3x + 3y &= -16 \end{aligned}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. Sometimes, judiciously choosing the order in which you apply elementary row operations enables you to avoid fractions.

REMARK

No matter which order you use, the reduced row-echelon form will be the same.

DISCOVERY

1. Without doing any row operations, explain why the following system of linear equations is consistent.

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ -5x_1 + 6x_2 - 17x_3 &= 0 \\ 7x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

2. The following system has more variables than equations. Why does it have an infinite number of solutions?

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 &= 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with infinitely many solutions.

EXAMPLE 8 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}.$$

Using a graphing utility, a software program, or Gauss-Jordan elimination, verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter t to represent x_3, you have

$$x_1 = 2 - 5t, \quad x_2 = -1 + 3t, \quad x_3 = t, \quad \text{where } t \text{ is any real number.}$$

Note that in Example 8 an arbitrary parameter was assigned to the *nonleading* variable x_3. You subsequently solved for the leading variables x_1 and x_2 as functions of t.

You have looked at two elimination methods for solving a system of linear equations. Which is better? To some degree the answer depends on personal preference. In real-life applications of linear algebra, systems of linear equations are usually solved by computer. Most computer programs use a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data. Because the examples and exercises in this text focus on the underlying concepts, you should know both elimination methods.

HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations in which each of the constant terms is zero are called **homogeneous.** A homogeneous system of m equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

A homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called **trivial** (or **obvious**).

REMARK

A homogeneous system of three equations in the three variables x_1, x_2, and x_3 must have $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ as a trivial solution.

EXAMPLE 9 Solving a Homogeneous System of Linear Equations

Solve the system of linear equations.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

SOLUTION

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

yields the following.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \qquad R_2 + (-2)R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad \left(\tfrac{1}{3}\right)R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad R_1 + R_2 \to R_1$$

The system of equations corresponding to this matrix is

$$\begin{aligned} x_1 \qquad + 2x_3 &= 0 \\ x_2 - \ x_3 &= 0. \end{aligned}$$

Using the parameter $t = x_3$, the solution set is $x_1 = -2t$, $x_2 = t$, and $x_3 = t$, where t is any real number.

This system has infinitely many solutions, one of which is the trivial solution ($t = 0$).

As illustrated in Example 9, a homogeneous system with fewer equations than variables has infinitely many solutions.

THEOREM 1.1 The Number of Solutions of a Homogeneous System

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

To prove Theorem 1.1, use the procedure in Example 9, but for a general matrix.

Seventh Edition

1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Matrix Size In Exercises 1–6, determine the size of the matrix.

1. $\begin{bmatrix} 1 & 2 & -4 \\ 3 & -4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & -1 & 1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$ **4.** $[-1]$

5. $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 3 & 4 & -10 \end{bmatrix}$

Elementary Row Operations In Exercises 7–10, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.

Original Matrix / *New Row-Equivalent Matrix*

7. $\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$ $\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$

Original Matrix / *New Row-Equivalent Matrix*

8. $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$ $\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$

Original Matrix / *New Row-Equivalent Matrix*

9. $\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$

Original Matrix / *New Row-Equivalent Matrix*

10. $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$ $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

Augmented Matrix In Exercises 11–18, find the solution set of the system of linear equations represented by the augmented matrix.

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ **16.** $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Row-Echelon Form In Exercises 19–24, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

19. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

System of Linear Equations In Exercises 25–38, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

25. $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$ **26.** $\begin{aligned} 2x + 6y &= 16 \\ -2x - 6y &= -16 \end{aligned}$

27. $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

28. $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

29. $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

30. $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$

31. $$\begin{aligned} x_1 \qquad\quad - 3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$$

32. $$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 24 \\ 2x_2 - x_3 &= 14 \\ 7x_1 - 5x_2 \qquad &= 6 \end{aligned}$$

33. $$\begin{aligned} 2x_1 + \qquad 3x_3 &= 3 \\ 4x_1 - 3x_2 + 7x_3 &= 5 \\ 8x_1 - 9x_2 + 15x_3 &= 10 \end{aligned}$$

34. $$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

35. $$\begin{aligned} 4x + 12y - 7z - 20w &= 22 \\ 3x + 9y - 5z - 28w &= 30 \end{aligned}$$

36. $$\begin{aligned} x + 2y + z &= 8 \\ -3x - 6y - 3z &= -21 \end{aligned}$$

37. $$\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$$

38. $$\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y \qquad + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$$

System of Linear Equations **In Exercises 39 and 40, use a software program or a graphing utility to solve the system of linear equations.**

39. $$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 6 \\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 &= 14 \\ x_2 - x_3 - x_4 - 3x_5 &= -3 \\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 &= 10 \\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 &= 13 \end{aligned}$$

40. $$\begin{aligned} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 + 3x_6 &= 0 \\ 2x_1 - x_2 + 3x_3 + x_4 - 3x_5 + 2x_6 &= 17 \\ x_1 + 3x_2 - 2x_3 + x_4 - 2x_5 - 3x_6 &= -5 \\ 3x_1 - 2x_2 + x_3 - x_4 + 3x_5 - 2x_6 &= -1 \\ -x_1 - 2x_2 + x_3 + 2x_4 - 2x_5 + 3x_6 &= 10 \\ x_1 - 3x_2 + x_3 + 3x_4 - 2x_5 + x_6 &= 11 \end{aligned}$$

Homogeneous System **In Exercises 41–44, solve the homogeneous linear system corresponding to the given coefficient matrix.**

41. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

44. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

45. **Finance** A small software corporation borrowed \$500,000 to expand its software line. The corporation borrowed some of the money at 9%, some at 10%, and some at 12%. Use a system of equations to determine how much was borrowed at each rate if the annual interest was \$52,000 and the amount borrowed at 10% was $2\frac{1}{2}$ times the amount borrowed at 9%. Solve the system using matrices.

46. **Tips** A food server examines the amount of money earned in tips after working an 8-hour shift. The server has a total of \$95 in denominations of \$1, \$5, \$10, and \$20 bills. The total number of paper bills is 26. The number of \$5 bills is 4 times the number of \$10 bills, and the number of \$1 bills is 1 less than twice the number of \$5 bills. Write a system of linear equations to represent the situation. Then use matrices to find the number of each denomination.

Matrix Representation **In Exercises 47 and 48, assume that the matrix is the *augmented* matrix of a system of linear equations, and (a) determine the number of equations and the number of variables, and (b) find the value(s) of k such that the system is consistent. Then assume that the matrix is the *coefficient* matrix of a *homogeneous* system of linear equations, and repeat parts (a) and (b).**

47. $A = \begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$

48. $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & k \\ 4 & -2 & 6 \end{bmatrix}$

Coefficient Design **In Exercises 49 and 50, find values of a, b, and c (if possible) such that the system of linear equations has (a) a unique solution, (b) no solution, and (c) infinitely many solutions.**

49. $$\begin{aligned} x + y \qquad &= 2 \\ y + z &= 2 \\ x \qquad + z &= 2 \\ ax + by + cz &= 0 \end{aligned}$$

50. $$\begin{aligned} x + y \qquad &= 0 \\ y + z &= 0 \\ x \qquad + z &= 0 \\ ax + by + cz &= 0 \end{aligned}$$

51. The following system has one solution: $x = 1$, $y = -1$, and $z = 2$.

$$\begin{aligned} 4x - 2y + 5z &= 16 && \text{Equation 1} \\ x + y \quad &= 0 && \text{Equation 2} \\ -x - 3y + 2z &= 6 && \text{Equation 3} \end{aligned}$$

Solve the systems provided by (a) Equations 1 and 2, (b) Equations 1 and 3, and (c) Equations 2 and 3. (d) How many solutions does each of these systems have?

52. Assume the system below has a unique solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 && \text{Equation 1} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 && \text{Equation 2} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 && \text{Equation 3} \end{aligned}$$

Does the system composed of Equations 1 and 2 have a unique solution, no solution, or infinitely many solutions?

Row Equivalence **In Exercises 53 and 54, find the reduced row-echelon matrix that is row-equivalent to the given matrix.**

53. $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ **54.** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

55. Writing Describe all possible 2×2 reduced row-echelon matrices. Support your answer with examples.

56. Writing Describe all possible 3×3 reduced row-echelon matrices. Support your answer with examples.

True or False? **In Exercises 57 and 58, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

57. (a) A 6×3 matrix has six rows.

(b) Every matrix is row-equivalent to a matrix in row-echelon form.

(c) If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1 \ 0 \ 0 \ 0 \ 0]$, then the original system is inconsistent.

(d) A homogeneous system of four linear equations in six variables has infinitely many solutions.

58. (a) A 4×7 matrix has four columns.

(b) Every matrix has a unique reduced row-echelon form.

(c) A homogeneous system of four linear equations in four variables is always consistent.

(d) Multiplying a row of a matrix by a constant is one of the elementary row operations.

59. Writing Is it possible for a system of linear equations with fewer equations than variables to have no solution? If so, give an example.

60. Writing Does a matrix have a unique row-echelon form? Illustrate your answer with examples. Is the reduced row-echelon form unique?

Row Equivalence **In Exercises 61 and 62, determine conditions on a, b, c, and d such that the matrix**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

will be row-equivalent to the given matrix.

61. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **62.** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Homogeneous System **In Exercises 63 and 64, find all values of λ (the Greek letter lambda) for which the homogeneous linear system has nontrivial solutions.**

63. $\begin{aligned} (\lambda - 2)x + \quad\quad y &= 0 \\ x + (\lambda - 2)y &= 0 \end{aligned}$

64. $\begin{aligned} (\lambda - 1)x + 2y &= 0 \\ x + \lambda y &= 0 \end{aligned}$

65. Writing Consider the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Perform the sequence of row operations.

(a) Add (-1) times the second row to the first row.

(b) Add 1 times the first row to the second row.

(c) Add (-1) times the second row to the first row.

(d) Multiply the first row by (-1).

What happened to the original matrix? Describe, in general, how to interchange two rows of a matrix using only the second and third elementary row operations.

66. The augmented matrix represents a system of linear equations that has been reduced using Gauss-Jordan elimination. Write a system of equations with nonzero coefficients that the reduced matrix could represent.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are many correct answers.

67. Writing Describe the row-echelon form of an augmented matrix that corresponds to a linear system that (a) is inconsistent, and (b) has infinitely many solutions.

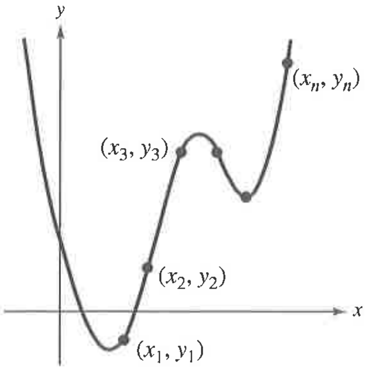
68. CAPSTONE In your own words, describe the difference between a matrix in row-echelon form and a matrix in reduced row-echelon form. Include an example of each to support your explanation.

1.3 Applications of Systems of Linear Equations

- Set up and solve a system of equations to fit a polynomial function to a set of data points.
- Set up and solve a system of equations to represent a network.

Systems of linear equations arise in a wide variety of applications. In this section you will look at two applications, and you will see more in subsequent chapters. The first application shows how to fit a polynomial function to a set of data points in the plane. The second application focuses on networks and Kirchhoff's Laws for electricity.

POLYNOMIAL CURVE FITTING



Polynomial Curve Fitting

Figure 1.4

Suppose n points in the xy-plane

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

represent a collection of data and you are asked to find a polynomial function of degree $n - 1$

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

whose graph passes through the specified points. This procedure is called **polynomial curve fitting.** If all x-coordinates of the points are distinct, then there is precisely one polynomial function of degree $n - 1$ (or less) that fits the n points, as shown in Figure 1.4.

To solve for the n coefficients of $p(x)$, substitute each of the n points into the polynomial function and obtain n linear equations in n variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$\begin{aligned}
a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_{n-1}x_1^{n-1} &= y_1 \\
a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_{n-1}x_2^{n-1} &= y_2 \\
&\vdots \\
a_0 + a_1x_n + a_2x_n^2 + \cdots + a_{n-1}x_n^{n-1} &= y_n
\end{aligned}$$

Example 1 demonstrates this procedure with a second-degree polynomial.

EXAMPLE 1 Polynomial Curve Fitting

Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points (1, 4), (2, 0), and (3, 12).

SOLUTION

Substituting $x = 1$, 2, and 3 into $p(x)$ and equating the results to the respective y-values produces the system of linear equations in the variables a_0, a_1, and a_2 shown below.

$$\begin{aligned}
p(1) &= a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\
p(2) &= a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\
p(3) &= a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12
\end{aligned}$$

The solution of this system is

$$a_0 = 24,\ a_1 = -28, \text{ and } a_2 = 8$$

so the polynomial function is

$$p(x) = 24 - 28x + 8x^2.$$

Figure 1.5 shows the graph of p.

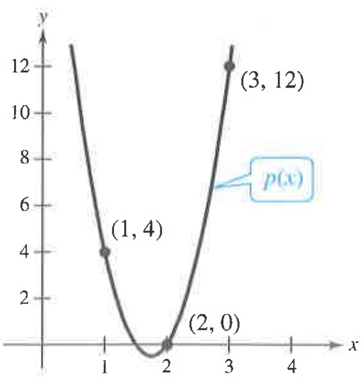


Figure 1.5

Simulation
Explore this concept further with an electronic simulation available at *www.cengagebrain.com.*

EXAMPLE 2 Polynomial Curve Fitting

Find a polynomial that fits the points

$(-2, 3), (-1, 5), (0, 1), (1, 4)$, and $(2, 10)$.

SOLUTION

Because you are given five points, choose a fourth-degree polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4.$$

Substituting the given points into $p(x)$ produces the following system of linear equations.

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\ a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10 \end{aligned}$$

The solution of these equations is

$$a_0 = 1, \quad a_1 = -\tfrac{30}{24}, \quad a_2 = \tfrac{101}{24}, \quad a_3 = \tfrac{18}{24}, \quad a_4 = -\tfrac{17}{24}$$

which means the polynomial function is

$$\begin{aligned} p(x) &= 1 - \tfrac{30}{24}x + \tfrac{101}{24}x^2 + \tfrac{18}{24}x^3 - \tfrac{17}{24}x^4 \\ &= \tfrac{1}{24}(24 - 30x + 101x^2 + 18x^3 - 17x^4). \end{aligned}$$

Figure 1.6 shows the graph of p.

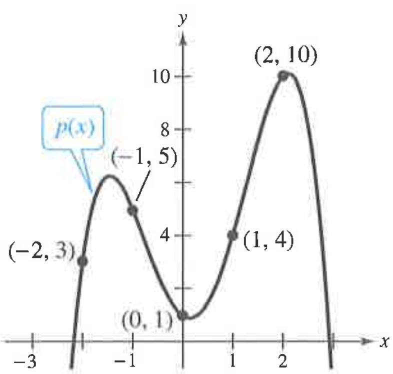


Figure 1.6

The system of linear equations in Example 2 is relatively easy to solve because the x-values are small. For a set of points with large x-values, it is usually best to *translate* the values before attempting the curve-fitting procedure. The next example demonstrates this approach.

EXAMPLE 3 Translating Large x-Values Before Curve Fitting

Find a polynomial that fits the points

$$\underbrace{(2006, 3)}_{(x_1, y_1)}, \quad \underbrace{(2007, 5)}_{(x_2, y_2)}, \quad \underbrace{(2008, 1)}_{(x_3, y_3)}, \quad \underbrace{(2009, 4)}_{(x_4, y_4)}, \quad \underbrace{(2010, 10)}_{(x_5, y_5)}.$$

SOLUTION

Because the given x-values are large, use the translation $z = x - 2008$ to obtain

$$\underbrace{(-2, 3)}_{(z_1, y_1)}, \quad \underbrace{(-1, 5)}_{(z_2, y_2)}, \quad \underbrace{(0, 1)}_{(z_3, y_3)}, \quad \underbrace{(1, 4)}_{(z_4, y_4)}, \quad \underbrace{(2, 10)}_{(z_5, y_5)}.$$

This is the same set of points as in Example 2. So, the polynomial that fits these points is

$$\begin{aligned} p(z) &= \tfrac{1}{24}(24 - 30z + 101z^2 + 18z^3 - 17z^4) \\ &= 1 - \tfrac{5}{4}z + \tfrac{101}{24}z^2 + \tfrac{3}{4}z^3 - \tfrac{17}{24}z^4. \end{aligned}$$

Letting $z = x - 2008$, you have

$$p(x) = 1 - \tfrac{5}{4}(x - 2008) + \tfrac{101}{24}(x - 2008)^2 + \tfrac{3}{4}(x - 2008)^3 - \tfrac{17}{24}(x - 2008)^4.$$

EXAMPLE 4 An Application of Curve Fitting

Find a polynomial that relates the periods of the three planets that are closest to the Sun to their mean distances from the Sun, as shown in the table. Then test the accuracy of the fit by using the polynomial to calculate the period of Mars. (In the table, the mean distance is given in astronomical units, and the period is given in years.)

Planet	*Mercury*	*Venus*	*Earth*	*Mars*
Mean Distance	0.387	0.723	1.000	1.524
Period	0.241	0.615	1.000	1.881

SOLUTION

Begin by fitting a quadratic polynomial function

$$p(x) = a_0 + a_1x + a_2x^2$$

to the points

$$(0.387, 0.241), (0.723, 0.615), \text{ and } (1, 1).$$

The system of linear equations obtained by substituting these points into $p(x)$ is

$$\begin{aligned} a_0 + 0.387a_1 + (0.387)^2a_2 &= 0.241 \\ a_0 + 0.723a_1 + (0.723)^2a_2 &= 0.615 \\ a_0 + a_1 + a_2 &= 1. \end{aligned}$$

The approximate solution of the system is

$$a_0 \approx -0.0634, \quad a_1 \approx 0.6119, \quad a_2 \approx 0.4515$$

which means that an approximation of the polynomial function is

$$p(x) = -0.0634 + 0.6119x + 0.4515x^2.$$

Using $p(x)$ to evaluate the period of Mars produces

$$p(1.524) \approx 1.918 \text{ years.}$$

Note that the actual period of Mars is 1.881 years. Figure 1.7 compares the estimate with the actual period graphically.

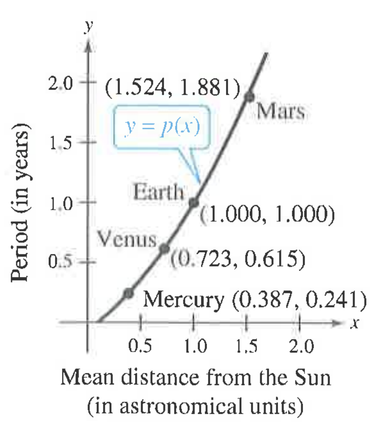


Figure 1.7

As illustrated in Example 4, a polynomial that fits some of the points in a data set is not necessarily an accurate model for other points in the data set. Generally, the farther the other points are from those used to fit the polynomial, the worse the fit. For instance, the mean distance of Jupiter from the Sun is 5.203 astronomical units. Using $p(x)$ in Example 4 to approximate the period gives 15.343 years—a poor estimate of Jupiter's actual period of 11.860 years.

The problem of curve fitting can be difficult. Types of functions other than polynomial functions may provide better fits. For instance, look again at the curve-fitting problem in Example 4. Taking the natural logarithms of the given distances and periods produces the following results.

Planet	*Mercury*	*Venus*	*Earth*	*Mars*
Mean Distance (x)	0.387	0.723	1.000	1.524
$\ln x$	−0.949	−0.324	0.0	0.421
Period (y)	0.241	0.615	1.000	1.881
$\ln y$	−1.423	−0.486	0.0	0.632

Now, fitting a polynomial to the logarithms of the distances and periods produces the *linear relationship*

$$\ln y = \tfrac{3}{2} \ln x$$

shown in Figure 1.8.

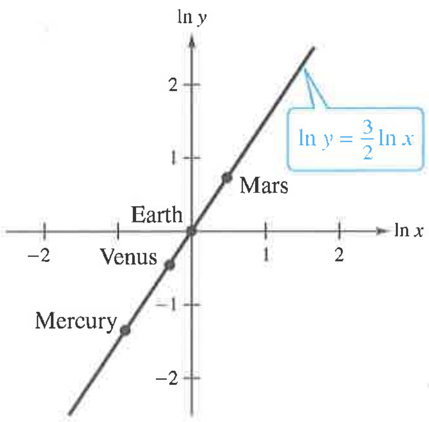


Figure 1.8

From $\ln y = \frac{3}{2} \ln x$, it follows that $y = x^{3/2}$, or $y^2 = x^3$. In other words, the square of the period (in years) of each planet is equal to the cube of its mean distance (in astronomical units) from the Sun. Johannes Kepler first discovered this relationship in 1619.

LINEAR ALGEBRA APPLIED

Researchers in Italy studying the acoustical noise levels from vehicular traffic at a busy three-way intersection on a college campus used a system of linear equations to model the traffic flow at the intersection. To help formulate the system of equations, "operators" stationed themselves at various locations along the intersection and counted the numbers of vehicles going by. (Source: *Acoustical Noise Analysis in Road Intersections: A Case Study*, Guarnaccia, Claudio, *Recent Advances in Acoustics & Music, Proceedings of the 11th WSEAS International Conference on Acoustics & Music: Theory & Applications*, June, 2010)

gemphotography/Shutterstock.com

NETWORK ANALYSIS

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering. In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction. For instance, the junction shown in Figure 1.9 has 25 units flowing into it, so there must be 25 units flowing out of it. You can represent this with the linear equation

$$x_1 + x_2 = 25.$$

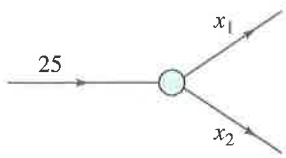


Figure 1.9

Because each junction in a network gives rise to a linear equation, you can analyze the flow through a network composed of several junctions by solving a system of linear equations. Example 5 illustrates this procedure.

EXAMPLE 5 Analysis of a Network

Set up a system of linear equations to represent the network shown in Figure 1.10. Then solve the system.

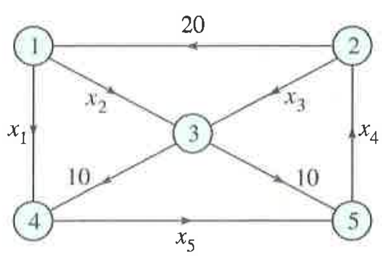


Figure 1.10

SOLUTION

Each of the network's five junctions gives rise to a linear equation, as follows.

$$\begin{array}{rcll}
x_1 + x_2 \qquad\qquad\qquad & = & 20 & \text{Junction 1} \\
x_3 - x_4 \qquad & = & -20 & \text{Junction 2} \\
x_2 + x_3 \qquad\qquad & = & 20 & \text{Junction 3} \\
x_1 \qquad\qquad\qquad - x_5 & = & -10 & \text{Junction 4} \\
-x_4 + x_5 & = & -10 & \text{Junction 5}
\end{array}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{bmatrix}.$$

Gauss-Jordan elimination produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the matrix above, you can see that

$$x_1 - x_5 = -10, \quad x_2 + x_5 = 30, \quad x_3 - x_5 = -10, \quad \text{and} \quad x_4 - x_5 = 10.$$

Letting $t = x_5$, you have

$$x_1 = t - 10, \quad x_2 = -t + 30, \quad x_3 = t - 10, \quad x_4 = t + 10, \quad x_5 = t$$

where t is any real number, so this system has infinitely many solutions.

In Example 5, suppose you could control the amount of flow along the branch labeled x_5. Using the solution of Example 5, you could then control the flow represented by each of the other variables. For instance, letting $t = 10$ would reduce the flow of x_1 and x_3 to zero, as shown in Figure 1.11.

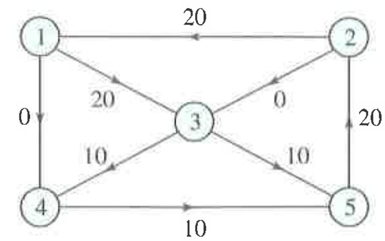


Figure 1.11

You may be able to see how the type of network analysis demonstrated in Example 5 could be used in problems dealing with the flow of traffic through the streets of a city or the flow of water through an irrigation system.

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

REMARK

A closed path is a sequence of branches such that the beginning point of the first branch coincides with the end point of the last branch.

1. All the current flowing into a junction must flow out of it.

2. The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

In an electrical network, current is measured in amperes, or amps (A), resistance is measured in ohms (Ω), and the product of current and resistance is measured in volts (V). The symbol —|⊢— represents a battery. The larger vertical bar denotes where the current flows out of the terminal. The symbol -\/\/\/- denotes resistance. An arrow in the branch indicates the direction of the current.

EXAMPLE 6 Analysis of an Electrical Network

Determine the currents I_1, I_2, and I_3 for the electrical network shown in Figure 1.12.

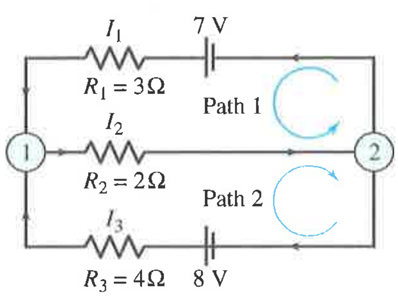


Figure 1.12

SOLUTION

Applying Kirchhoff's first law to either junction produces

$$I_1 + I_3 = I_2 \qquad \text{Junction 1 or Junction 2}$$

and applying Kirchhoff's second law to the two paths produces

$$\begin{aligned} R_1I_1 + R_2I_2 &= 3I_1 + 2I_2 = 7 && \text{Path 1} \\ R_2I_2 + R_3I_3 &= 2I_2 + 4I_3 = 8. && \text{Path 2} \end{aligned}$$

So, you have the following system of three linear equations in the variables I_1, I_2, and I_3.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 3I_1 + 2I_2 &= 7 \\ 2I_2 + 4I_3 &= 8 \end{aligned}$$

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

produces the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which means $I_1 = 1$ amp, $I_2 = 2$ amps, and $I_3 = 1$ amp.

EXAMPLE 7 Analysis of an Electrical Network

Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for the electrical network shown in Figure 1.13.

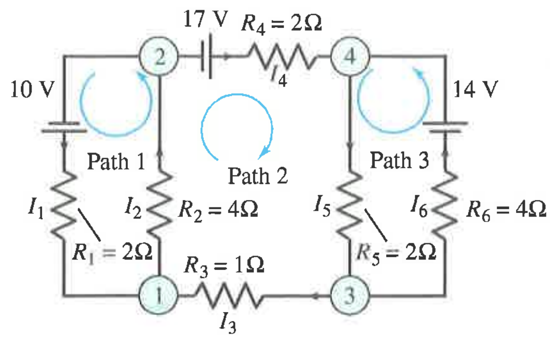


Figure 1.13

SOLUTION

Applying Kirchhoff's first law to the four junctions produces

$$
\begin{aligned}
I_1 + I_3 &= I_2 && \text{Junction 1}\\
I_1 + I_4 &= I_2 && \text{Junction 2}\\
I_3 + I_6 &= I_5 && \text{Junction 3}\\
I_4 + I_6 &= I_5 && \text{Junction 4}
\end{aligned}
$$

and applying Kirchhoff's second law to the three paths produces

$$
\begin{aligned}
2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10 && \text{Path 1}\\
4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17 && \text{Path 2}\\
2I_5 + 4I_6 &= 14. && \text{Path 3}
\end{aligned}
$$

You now have the following system of seven linear equations in the variables I_1, I_2, I_3, I_4, I_5, and I_6.

$$
\begin{aligned}
I_1 - I_2 + I_3 \qquad\qquad\qquad &= 0\\
I_1 - I_2 \qquad + I_4 \qquad\qquad &= 0\\
I_3 \qquad - I_5 + I_6 &= 0\\
I_4 - I_5 + I_6 &= 0\\
2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10\\
4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17\\
2I_5 + 4I_6 &= 14
\end{aligned}
$$

The augmented matrix for this system is

$$
\begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 1 & -1 & 1 & 0\\
2 & 4 & 0 & 0 & 0 & 0 & 10\\
0 & 4 & 1 & 2 & 2 & 0 & 17\\
0 & 0 & 0 & 0 & 2 & 4 & 14
\end{bmatrix}.
$$

Using Gauss-Jordan elimination, a graphing utility, or a software program, solve this system to obtain

$$I_1 = 1, \quad I_2 = 2, \quad I_3 = 1, \quad I_4 = 1, \quad I_5 = 3, \quad \text{and} \quad I_6 = 2$$

meaning $I_1 = 1$ amp, $I_2 = 2$ amps, $I_3 = 1$ amp, $I_4 = 1$ amp, $I_5 = 3$ amps, and $I_6 = 2$ amps.

1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Polynomial Curve Fitting **In Exercises 1–12, (a) determine the polynomial function whose graph passes through the given points, and (b) sketch the graph of the polynomial function, showing the given points.**

1. $(2, 5), (3, 2), (4, 5)$

2. $(0, 0), (2, -2), (4, 0)$

3. $(2, 4), (3, 6), (5, 10)$

4. $(2, 4), (3, 4), (4, 4)$

5. $(-1, 3), (0, 0), (1, 1), (4, 58)$

6. $(0, 42), (1, 0), (2, -40), (3, -72)$

7. $(-2, 28), (-1, 0), (0, -6), (1, -8), (2, 0)$

8. $(-4, 18), (0, 1), (4, 0), (6, 28), (8, 135)$

9. $(2006, 5), (2007, 7), (2008, 12)$

10. $(2005, 150), (2006, 180), (2007, 240), (2008, 360)$

11. $(0.072, 0.203), (0.120, 0.238), (0.148, 0.284)$

12. $(1, 1), (1.189, 1.587), (1.316, 2.080), (1.414, 2.520)$

13. Use $\sin 0 = 0$, $\sin\left(\frac{\pi}{2}\right) = 1$, and $\sin \pi = 0$ to estimate $\sin\left(\frac{\pi}{3}\right)$.

14. Use $\log_2 1 = 0$, $\log_2 2 = 1$, and $\log_2 4 = 2$ to estimate $\log_2 3$.

15. Find an equation of the circle passing through the points $(1, 3)$, $(-2, 6)$, and $(4, 2)$.

16. Find an equation of the ellipse passing through the points $(-5, 1)$, $(-3, 2)$, $(-1, 1)$, and $(-3, 0)$.

17. **Population** The U.S. census lists the population of the United States as 249 million in 1990, 281 million in 2000, and 309 million in 2010. Fit a second-degree polynomial passing through these three points and use it to predict the population in 2020 and in 2030. (Source: U.S. Census Bureau)

18. **Population** The table shows the U.S. population figures for the years 1940, 1950, 1960, and 1970. (Source: U.S. Census Bureau)

Year	1940	1950	1960	1970
Population (in millions)	132	151	179	203

(a) Find a cubic polynomial that fits these data and use it to estimate the population in 1980.

(b) The actual population in 1980 was 227 million. How does your estimate compare?

19. **Net Profit** The table shows the net profits (in millions of dollars) for Microsoft from 2003 through 2010. (Source: Microsoft Corp.)

Year	2003	2004	2005	2006
Net Profit	10,526	11,330	12,715	12,599

Year	2007	2008	2009	2010
Net Profit	14,065	17,681	14,569	18,760

(a) Set up a system of equations to fit the data for the years 2003, 2004, 2005, and 2006 to a cubic model.

(b) Solve the system. Does the solution produce a reasonable model for determining net profits after 2006? Explain.

20. **Sales** The table shows the sales (in billions of dollars) for Wal-Mart stores from 2002 through 2009. (Source: Wal-Mart Stores, Inc.)

Year	2002	2003	2004	2005
Sales	244.5	256.3	285.2	312.4

Year	2006	2007	2008	2009
Sales	345.0	374.5	401.2	405.0

(a) Set up a system of equations to fit the data for the years 2002, 2003, 2004, 2005, and 2006 to a quartic model.

(b) Solve the system. Does the solution produce a reasonable model for determining sales after 2006? Explain.

21. **Guided Proof** Prove that if a polynomial function $p(x) = a_0 + a_1x + a_2x^2$ is zero for $x = -1$, $x = 0$, and $x = 1$, then $a_0 = a_1 = a_2 = 0$.

Getting Started: Write a system of linear equations and solve the system for a_0, a_1, and a_2.

(i) Substitute $x = -1$, 0, and 1 into $p(x)$.

(ii) Set the result equal to 0.

(iii) Solve the resulting system of linear equations in the variables a_0, a_1, and a_2.

22. Generalizing the statement in Exercise 21, if a polynomial function

$$p(x) = a_0 + a_1x + \cdot\cdot\cdot + a_{n-1}x^{n-1}$$

is zero for more than $n - 1$ x-values, then

$$a_0 = a_1 = \cdot\cdot\cdot = a_{n-1} = 0.$$

Use this result to prove that there is at most one polynomial function of degree $n - 1$ (or less) whose graph passes through n points in the plane with distinct x-coordinates.

23. Calculus The graph of a cubic polynomial function has horizontal tangents at $(1, -2)$ and $(-1, 2)$. Find an equation for the cubic and sketch its graph.

24. Calculus The graph of a parabola passes through the points $(0, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ and has a horizontal tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Find an equation for the parabola and sketch its graph.

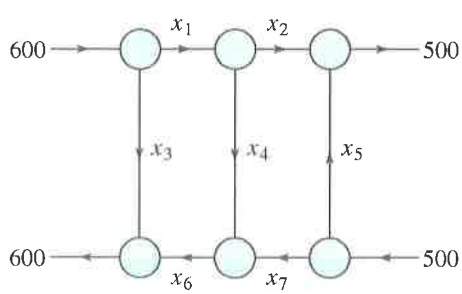
25. (a) The graph of a function f passes through the points $(0, 1)$, $\left(2, \frac{1}{3}\right)$, and $\left(4, \frac{1}{5}\right)$. Find a quadratic function whose graph passes through these points.

(b) Find a polynomial function p of degree 2 or less that passes through the points $(0, 1)$, $(2, 3)$, and $(4, 5)$. Then sketch the graph of $y = 1/p(x)$ and compare this graph with the graph of the polynomial function found in part (a).

26. Writing Try to fit the graph of a polynomial function to the values shown in the table. What happens, and why?

x	1	2	3	3	4
y	1	1	2	3	4

27. Network Analysis Water is flowing through a network of pipes (in thousands of cubic meters per hour), as shown in the figure.

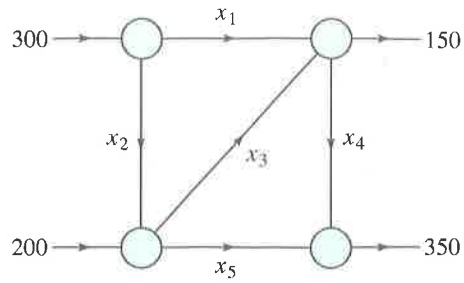


(a) Solve this system for the water flow represented by x_i, $i = 1, 2, \ldots, 7$.

(b) Find the water flow when $x_6 = x_7 = 0$.

(c) Find the water flow when $x_5 = 1000$ and $x_6 = 0$.

28. Network Analysis The figure shows the flow of traffic (in vehicles per hour) through a network of streets.

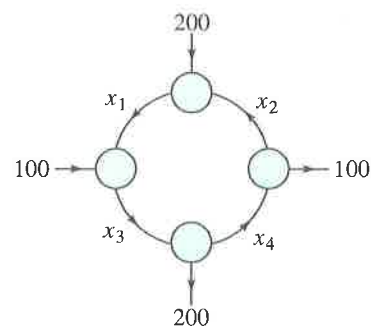


(a) Solve this system for x_i, $i = 1, 2, \ldots, 5$.

(b) Find the traffic flow when $x_2 = 200$ and $x_3 = 50$.

(c) Find the traffic flow when $x_2 = 150$ and $x_3 = 0$.

29. Network Analysis The figure shows the flow of traffic (in vehicles per hour) through a network of streets.

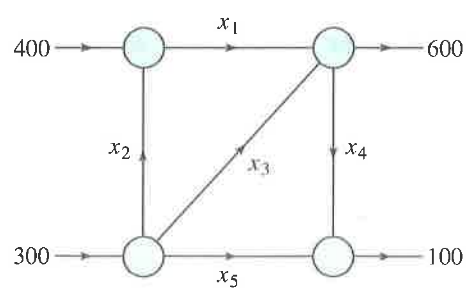


(a) Solve this system for x_i, $i = 1, 2, 3, 4$.

(b) Find the traffic flow when $x_4 = 0$.

(c) Find the traffic flow when $x_4 = 100$.

30. Network Analysis The figure shows the flow of traffic through a network of streets.

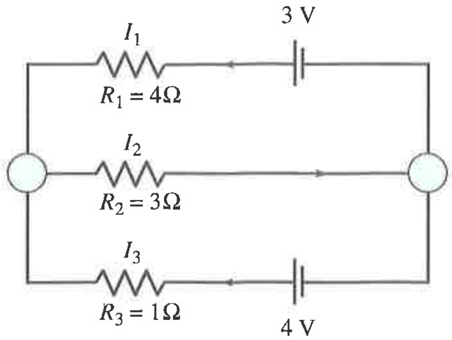


(a) Solve this system for x_i, $i = 1, 2, \ldots, 5$.

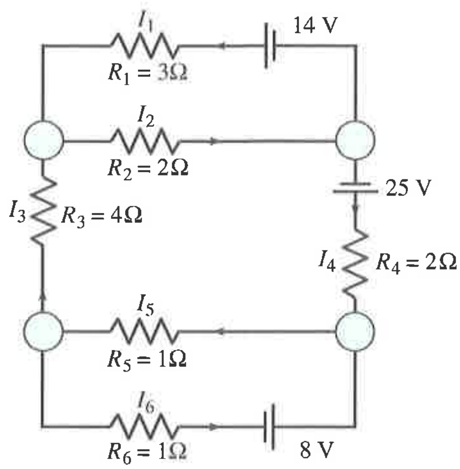
(b) Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.

(c) Find the traffic flow when $x_3 = x_5 = 100$.

31. Network Analysis Determine the currents I_1, I_2, and I_3 for the electrical network shown in the figure.



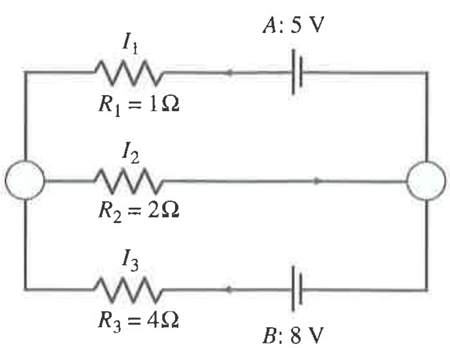
32. Network Analysis Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for the electrical network shown in the figure.



33. Network Analysis

(a) Determine the currents I_1, I_2, and I_3 for the electrical network shown in the figure.

(b) How is the result affected when A is changed to 2 volts and B is changed to 6 volts?



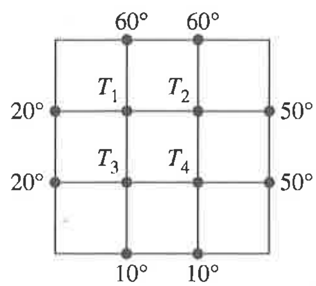
34. CAPSTONE

(a) Explain how to use systems of linear equations for polynomial curve fitting.

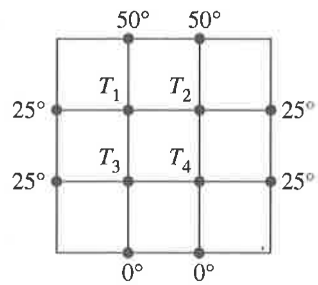
(b) Explain how to use systems of linear equations to perform network analysis.

Temperature **In Exercises 35 and 36, the figure shows the boundary temperatures (in degrees Celsius) of an insulated thin metal plate. The steady-state temperature at an interior junction is approximately equal to the mean of the temperatures at the four surrounding junctions. Use a system of linear equations to approximate the interior temperatures T_1, T_2, T_3, and T_4.**

35.



36.



Partial Fraction Decomposition **In Exercises 37 and 38, use a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.**

37. $\dfrac{4x^2}{(x+1)^2(x-1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

38. $\dfrac{3x^2-7x-12}{(x+4)(x-4)^2} = \dfrac{A}{x+4} + \dfrac{B}{x-4} + \dfrac{C}{(x-4)^2}$

Calculus **In Exercises 39 and 40, find the values of x, y, and λ that satisfy the system of equations. Such systems arise in certain problems of calculus, and λ is called the Lagrange multiplier.**

39.
$$\begin{aligned} 2x \quad\quad + \lambda \quad &= 0 \\ 2y + \lambda \quad &= 0 \\ x + y \quad\quad - 4 &= 0 \end{aligned}$$

40.
$$\begin{aligned} 2y + 2\lambda + 2 &= 0 \\ 2x \quad\quad + \lambda + 1 &= 0 \\ 2x + y \quad\quad - 100 &= 0 \end{aligned}$$

1 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Linear Equations **In Exercises 1–6, determine whether the equation is linear in the variables x and y.**

1. $2x - y^2 = 4$
2. $2xy - 6y = 0$
3. $(\sin \pi)x + y = 2$
4. $e^{-2}x + 5y = 8$
5. $\frac{2}{x} + 4y = 3$
6. $\frac{x}{2} - \frac{y}{4} = 0$

Parametric Representation **In Exercises 7 and 8, find a parametric representation of the solution set of the linear equation.**

7. $-4x + 2y - 6z = 1$
8. $3x_1 + 2x_2 - 4x_3 = 0$

System of Linear Equations **In Exercises 9–20, solve the system of linear equations.**

9. $\begin{aligned} x + y &= 2 \\ 3x - y &= 0 \end{aligned}$

10. $\begin{aligned} x + y &= -1 \\ 3x + 2y &= 0 \end{aligned}$

11. $\begin{aligned} 3y &= 2x \\ y &= x + 4 \end{aligned}$

12. $\begin{aligned} x &= y + 3 \\ 4x &= y + 10 \end{aligned}$

13. $\begin{aligned} y + x &= 0 \\ 2x + y &= 0 \end{aligned}$

14. $\begin{aligned} y &= -4x \\ y &= x \end{aligned}$

15. $\begin{aligned} x - y &= 9 \\ -x + y &= 1 \end{aligned}$

16. $\begin{aligned} 40x_1 + 30x_2 &= 24 \\ 20x_1 + 15x_2 &= -14 \end{aligned}$

17. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 0 \\ 3x + 2(y + 5) &= 10 \end{aligned}$

18. $\begin{aligned} \tfrac{1}{3}x + \tfrac{4}{7}y &= 3 \\ 2x + 3y &= 15 \end{aligned}$

19. $\begin{aligned} 0.2x_1 + 0.3x_2 &= 0.14 \\ 0.4x_1 + 0.5x_2 &= 0.20 \end{aligned}$

20. $\begin{aligned} 0.2x - 0.1y &= 0.07 \\ 0.4x - 0.5y &= -0.01 \end{aligned}$

Matrix Size **In Exercises 21 and 22, determine the size of the matrix.**

21. $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 1 \\ -4 & -1 \\ 0 & 5 \end{bmatrix}$

Augmented Matrix **In Exercises 23 and 24, find the solution set of the system of linear equations represented by the augmented matrix.**

23. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row-Echelon Form **In Exercises 25–28, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.**

25. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27. $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

System of Linear Equations **In Exercises 29–38, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.**

29. $\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$

30. $\begin{aligned} 2x + 3y + z &= 10 \\ 2x - 3y - 3z &= 22 \\ 4x - 2y + 3z &= -2 \end{aligned}$

31. $\begin{aligned} 2x + 3y + 3z &= 3 \\ 6x + 6y + 12z &= 13 \\ 12x + 9y - z &= 2 \end{aligned}$

32. $\begin{aligned} 2x + y + 2z &= 4 \\ 2x + 2y &= 5 \\ 2x - y + 6z &= 2 \end{aligned}$

33. $\begin{aligned} x - 2y + z &= -6 \\ 2x - 3y &= -7 \\ -x + 3y - 3z &= 11 \end{aligned}$

34. $\begin{aligned} 2x + 6z &= -9 \\ 3x - 2y + 11z &= -16 \\ 3x - y + 7z &= -11 \end{aligned}$

35. $\begin{aligned} x + 2y + 6z &= 1 \\ 2x + 5y + 15z &= 4 \\ 3x + y + 3z &= -6 \end{aligned}$

36. $\begin{aligned} 2x_1 + 5x_2 - 19x_3 &= 34 \\ 3x_1 + 8x_2 - 31x_3 &= 54 \end{aligned}$

37. $\begin{aligned} 2x_1 + x_2 + x_3 + 2x_4 &= -1 \\ 5x_1 - 2x_2 + x_3 - 3x_4 &= 0 \\ -x_1 + 3x_2 + 2x_3 + 2x_4 &= 1 \\ 3x_1 + 2x_2 + 3x_3 - 5x_4 &= 12 \end{aligned}$

38. $\begin{aligned} x_1 + 5x_2 + 3x_3 &= 14 \\ 4x_2 + 2x_3 + 5x_4 &= 3 \\ 3x_3 + 8x_4 + 6x_5 &= 16 \\ 2x_1 + 4x_2 - 2x_5 &= 0 \\ 2x_1 - x_3 &= 0 \end{aligned}$

System of Linear Equations **In Exercises 39–42, use a software program or a graphing utility to solve the system of linear equations.**

39. $$\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$$

40. $$\begin{aligned} 2x + 10y + 2z &= 6 \\ x + 5y + 2z &= 6 \\ x + 5y + z &= 3 \\ -3x - 15y + 3z &= -9 \end{aligned}$$

41. $$\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y \quad\quad + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$$

42. $$\begin{aligned} x + 2y + z + 3w &= 0 \\ x - y \quad\quad + w &= 0 \\ 5y - z + 2w &= 0 \end{aligned}$$

Homogeneous System **In Exercises 43–46, solve the homogeneous system of linear equations.**

43. $$\begin{aligned} x_1 - 2x_2 - 8x_3 &= 0 \\ 3x_1 + 2x_2 \quad\quad &= 0 \end{aligned}$$

44. $$\begin{aligned} 2x_1 + 4x_2 - 7x_3 &= 0 \\ x_1 - 3x_2 + 9x_3 &= 0 \end{aligned}$$

45. $$\begin{aligned} 2x_1 - 8x_2 + 4x_3 &= 0 \\ 3x_1 - 10x_2 + 7x_3 &= 0 \\ 10x_2 + 5x_3 &= 0 \end{aligned}$$

46. $$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 \\ x_1 + 4x_2 + \tfrac{1}{2}x_3 &= 0 \end{aligned}$$

47. Determine the values of k such that the system of linear equations is inconsistent.

$$\begin{aligned} kx + y &= 0 \\ x + ky &= 1 \end{aligned}$$

48. Determine the values of k such that the system of linear equations has exactly one solution.

$$\begin{aligned} x - y + 2z &= 0 \\ -x + y - z &= 0 \\ x + ky + z &= 0 \end{aligned}$$

49. Find values of a and b such that the system of linear equations has (a) no solution, (b) exactly one solution, and (c) infinitely many solutions.

$$\begin{aligned} x + 2y &= 3 \\ ax + by &= -9 \end{aligned}$$

50. Find (if possible) values of a, b, and c such that the system of linear equations has (a) no solution, (b) exactly one solution, and (c) infinitely many solutions.

$$\begin{aligned} 2x - y + z &= a \\ x + y + 2z &= b \\ 3y + 3z &= c \end{aligned}$$

51. Writing Describe a method for showing that two matrices are row-equivalent. Are the two matrices below row-equivalent?

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \\ 5 & 5 & 10 \end{bmatrix}$$

52. Writing Describe all possible 2×3 reduced row-echelon matrices. Support your answer with examples.

53. Let $n \geq 3$. Find the reduced row-echelon form of the $n \times n$ matrix.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \vdots & \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2 \end{bmatrix}$$

54. Find all values of λ for which the homogeneous system of linear equations has nontrivial solutions.

$$\begin{aligned} (\lambda + 2)x_1 - \quad\quad 2x_2 + 3x_3 &= 0 \\ -2x_1 + (\lambda - 1)x_2 + 6x_3 &= 0 \\ x_1 + \quad\quad 2x_2 + \lambda x_3 &= 0 \end{aligned}$$

True or False? **In Exercises 55 and 56, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

55. (a) There is only one way to parametrically represent the solution set of a linear equation.

(b) A consistent system of linear equations can have infinitely many solutions.

56. (a) A homogeneous system of linear equations must have at least one solution.

(b) A system of linear equations with fewer equations than variables always has at least one solution.

57. Sports In Super Bowl I, on January 15, 1967, the Green Bay Packers defeated the Kansas City Chiefs by a score of 35 to 10. The total points scored came from a combination of touchdowns, extra-point kicks, and field goals, worth 6, 1, and 3 points, respectively. The numbers of touchdowns and extra-point kicks were equal. There were six times as many touchdowns as field goals. Find the numbers of touchdowns, extra-point kicks, and field goals scored. (Source: National Football League)

58. Agriculture A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

Partial Fraction Decomposition **In Exercises 59 and 60, use a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.**

59. $\dfrac{8x^2}{(x-1)^2(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

60. $\dfrac{3x^2+3x-2}{(x+1)^2(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x+1)^2}$

Polynomial Curve Fitting **In Exercises 61 and 62, (a) determine the polynomial function whose graph passes through the given points, and (b) sketch the graph of the polynomial function, showing the given points.**

61. $(2, 5), (3, 0), (4, 20)$

62. $(-1, -1), (0, 0), (1, 1), (2, 4)$

63. Sales A company has sales (measured in millions) of \$50, \$60, and \$75 during three consecutive years. Find a quadratic function that fits these data, and use it to predict the sales during the fourth year.

64. The polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

is zero when $x = 1, 2, 3$, and 4. What are the values of a_0, a_1, a_2, and a_3?

65. Deer Population A wildlife management team studied the population of deer in one small tract of a wildlife preserve. The table shows the population and the number of years since the study began.

Year	0	4	80
Population	80	68	30

(a) Set up a system of equations to fit the data to a quadratic polynomial function.

(b) Solve the system.

(c) Use a graphing utility to fit the data to a quadratic model.

(d) Compare the quadratic polynomial function in part (b) with the model in part (c).

(e) Cite the statement from the text that verifies your results.

66. Vertical Motion An object moving vertically is at the given heights at the specified times. Find the position equation

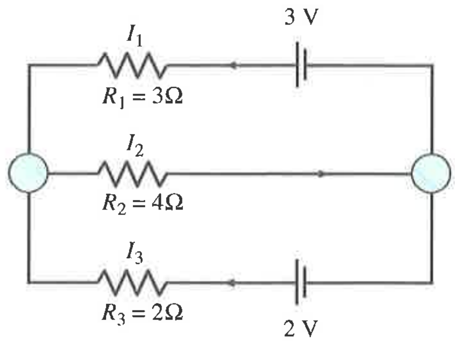
$$s = \tfrac{1}{2}at^2 + v_0t + s_0$$

for the object.

(a) At $t = 1$ second, $s = 134$ feet
At $t = 2$ seconds, $s = 86$ feet
At $t = 3$ seconds, $s = 6$ feet

(b) At $t = 1$ second, $s = 184$ feet
At $t = 2$ seconds, $s = 116$ feet
At $t = 3$ seconds, $s = 16$ feet

67. Network Analysis Determine the currents I_1, I_2, and I_3 for the electrical network shown in the figure.



68. Network Analysis The figure shows the flow through a network.

(a) Solve the system for x_i, $i = 1, 2, \ldots, 6$.

(b) Find the flow when $x_3 = 100$, $x_5 = 50$, and $x_6 = 50$.