MATH 211 Test #3 ♡ May 2, 2023

Name: Richard R. Hammack Score:______

Directions No calculators. Please put all phones, smart watches, etc., away.

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1. (16 points) This problem concerns the following statement.
   P: There is a subset X of $\mathbb{N}$ for which $X \cap \mathbb{N} = \emptyset$.

   (a) Is the statement P true or false? **Explain.**

   True! There does exist such a subset of $\mathbb{N}$, namely $X = \phi \subseteq \mathbb{N}$. For $X = \phi$ is a subset of $\mathbb{N}$ and $X \cap \mathbb{N} = \phi$.

   (b) Write the statement P in symbolic form.

   $\exists X \subseteq \mathbb{N}, X \cap \mathbb{N} = \phi$

   (c) Form the negation $\neg P$ of your answer from (b), and simplify.

   $\neg(\exists X \subseteq \mathbb{N}, X \cap \mathbb{N} = \phi) = \forall X \subseteq \mathbb{N}, \neg(X \cap \mathbb{N} = \phi)$
   $= \boxed{\forall X \subseteq \mathbb{N}, X \cap \mathbb{N} \neq \phi}$

   (d) Write the negation $\neg P$ as an English sentence. (The sentence may use mathematical symbols.)

   Every subset X of $\mathbb{N}$ has the property that $X \cap \mathbb{N} \neq \phi$.

2. (6 points) Complete the first and last lines of each of the following proof outlines.

Proposition: If P, then Q.	**Proposition:** If P, then Q.	**Proposition:** If P, then Q.
Proof: (Direct)	**Proof:** (Contrapositive)	**Proof:** (Contradiction)
Suppose P	Suppose $\neg Q$	Suppose $P \wedge \neg Q$
⋮	⋮	⋮
Therefore Q. ■	Therefore $\neg P$. ■	Therefore $C \wedge \neg C$. ■

3. (16 points) **Prove:** If $n \in \mathbb{Z}$, then $4|n^2$ or $4|(n^2+3)$. [Use direct proof, with cases]

Proof Suppose $n \in \mathbb{Z}$.

Case 1 Suppose n is even. Then $n = 2a$ for some $a \in \mathbb{N}$. Thus $n^2 = (2a)^2 = 4a^2$, so $n^2 = 4d$ for $d = a^2 \in \mathbb{N}$. By definition of divides, it follows that $4|n^2$.

Case 2 Suppose n is odd, so $n = 2a+1$ for some integer a. Then $n^2+3 = (2a+1)^2+3 = 4a^2+4a+1+3 = 4a^2+4a+4 = 4(a^2+a+1)$.
In other words $n^2+3 = 4d$, where $d = a^2+a+1$.
Consequently $4|(n^2+3)$.

By the above cases, either $4|n^2$ or $4(n^2+3)$. ∎

4. (16 points) Suppose $n \in \mathbb{Z}$. **Prove:** If n^2 is even, then n is even. [Use contrapositive.]

Proof Suppose n is not even.
Then n is odd, so $n = 2a+1$ for $a \in \mathbb{Z}$.
Therefore $n^2 = (2a+1)^2 = 4a^2+4a+1 = 2(2a^2+2a)+1$.
So $n^2 = 2d+1$, where $d = 2a^2+2a \in \mathbb{Z}$.
Consequently n^2 is odd, so n^2 is not even. ∎

5. (16 points) Suppose $a, b, c \in \mathbb{Z}$. **Prove:** If $a|b$ and $b|c$, then $a|c$. [Use any appropriate method.]

Proof (Direct) Suppose $a|b$ and $b|c$.
By definition of divides, this means
$b = ak$ and $c = bl$ for some $k, l \in \mathbb{Z}$.
Thus $c = bl = (ak)l = a(kl)$, so
$c = am$ for $m = kl \in \mathbb{Z}$.
Therefore $a|c$. $\blacksquare$

6. (15 points) Prove or disprove: If $a, b \in \mathbb{N}$, then $a + b < ab$.

FALSE

For a counterexample, take $a = 1$ and $b = 2$.

Then $a+b < ab$ is not true.

7. (15 points)Prove or disprove: Given $a, b, c \in \mathbb{Z}$, if $a|bc$, then $a|b$ or $a|c$.

FALSE

For a counterexample, take $a = 4$, $b = 2$ and $c = 6$

Then $a|bc$ but $a \nmid b$ and $a \nmid c$.