MATH 211 | Test #3 ♠ | December 5, 2019

Name: Key | R. Hammack | Score: 100

Directions No calculators. Please put all phones, etc., away.

1. (12 points) This problem concerns the following statement.
   P: There is a number $n \in \mathbb{Z}$ for which $m \mid n$ for every $m \in \mathbb{Z}$.

   (a) Is the statement P true or false? **Explain.**

   True There is a number $n \in \mathbb{Z}$ for which $m \mid n$ for any $m \in \mathbb{Z}$. That number is $n=0$. Given any $m \in \mathbb{Z}$, $0 = m \cdot 0$, so $n = m \cdot c$ for $c=0$ and hence $m \mid n$.

   (b) Write the statement P in symbolic form.

   $\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m \mid n$

   (c) Form the negation $\sim P$ of your answer from (b), and simplify.

   $\sim(\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m \mid n) =$

   $\boxed{\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m \nmid n}$

   (d) Write the negation $\sim P$ as an English sentence.
   (The sentence may use mathematical symbols.)

   For any $n \in \mathbb{Z}$, there is a number $m \in \mathbb{Z}$ for which $m \nmid n$.

2. (2 points) Complete the first and last lines of each of the following proof outlines.

Proposition: If P, then Q. **Proof:** (Direct) Suppose P ⋮ Therefore Q. ■	**Proposition:** If P, then Q. **Proof:** (Contrapositive) Suppose $\sim Q$ ⋮ Therefore $\sim P$. ■	**Proposition:** If P, then Q. **Proof:** (Contradiction) Suppose $P \wedge \sim Q$ ⋮ Therefore $C \wedge \sim C$. ■

3. (12 points) Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$.
Prove: If $a \equiv b \pmod{n}$, then $ab \equiv b^2 \pmod{n}$. [Use direct proof.]

Proof (Direct)

Suppose $a \equiv b \pmod{n}$, which means $n \mid (a-b)$.
Consequently there is an integer c for which

$$a - b = nc$$

Now multiply both sides of this equation by b.

$$(a-b)b = ncb$$

$$ab - b^2 = n(cb)$$

From this the definition of divides yields

$$n \mid (ab - b^2).$$

Therefore $ab \equiv b^2 \pmod{n}$. $\blacksquare$

4. (12 points) Suppose $a, b, c \in \mathbb{Z}$. **Prove:** If $a \nmid bc$, then $a \nmid b$ and $a \nmid c$. [Use contrapositive.]

Proof (Contrapositive)

Suppose that it is not true that $a \nmid b$ and $a \nmid c$.
Then $a \mid b$ or $a \mid c$.

CASE I Suppose $a \mid b$. Then $b = ak$ for some $k \in \mathbb{Z}$,
But $b = ak$ means $bc = a(kc)$ which means $a \mid bc$.

CASE II Suppose $a \mid c$. Then $c = a\ell$ for some $\ell \in \mathbb{Z}$
But $c = a\ell$ means $bc = a(\ell b)$, which means $a \mid bc$,

In either case we get $a \mid bc$.

Therefore it is not the case that $a \nmid bc$. $\blacksquare$

5. (12 points) **Prove:** If $4 \mid (a^2+b^2)$, then a and b are not both odd. [Use contradiction.]

Proof For the sake of contradiction, suppose $4 \mid (a^2+b^2)$ but it is not not the case that both a and b are not both odd, In other words $4 \mid (a^2+b^2)$ and a and b are both odd, Consequently

$$a^2+b^2 = 4k \quad \text{for some } k \in \mathbb{Z}$$

and $a = 2m+1$ and $b = 2n+1$ for $m, n \in \mathbb{Z}$. Plugging these into the above equation yields

$$(2m+1)^2 + (2n+1)^2 = 4k$$
$$4m^2+4m+1+4n^2+4n+1 = 4k$$
$$4m^2+4m+4n^2+4n+2 = 4k$$
$$2m^2+2m+2n^2+2n+1 = 2k$$
$$2(m^2+m+n^2+n)+1 = 2k$$

So an even number equals an odd number This is a contradiction ■

6. (12 points) Suppose $a, b, c \in \mathbb{Z}$. **Prove:** If $a \mid b$ and $a \mid (b+c)$, then $a \mid c$.

Proof (Direct) Suppose $a \mid b$ and $a \mid (b+c)$

Then $b = ak$ and $b+c = a\ell$ for $k, \ell \in \mathbb{Z}$.

Thus $b = ak$ and $b = a\ell - c$

Therefore $ak = a\ell - c$, and consequently

$c = a\ell - ak = a(\ell - k)$

Then $c = am$ for $m = \ell - k \in \mathbb{Z}$.

Therefore $a \mid c$. ■

7. (14 points) Suppose $n \in \mathbb{Z}$. Prove: n^2+3 is odd if and only if $n+2$ is even.

Proof:

($\Longrightarrow$) If n^2+3 is odd, then $n+2$ is even.

(contrapositive) Suppose $n+2$ is not even. Thus $n+2$ is odd, so $n+2 = 2k+1$ for $k \in \mathbb{Z}$, and thus $n = 2k-1$. Therefore

$$n^2+3 = (2k-1)^2+3 = 4k^2-4k+1+3$$
$$= 4k^2-4k+4 = 2(2k^2-2k+2)$$

Therefore n^3+3 is even, so it is not odd.

($\Longleftarrow$) If $n+2$ is even, then n^2+3 is odd,

(Direct) Suppose $n+2$ is even, so $n+2 = 2k$ for some $k \in \mathbb{Z}$, so $n = 2k-2$.

Consequently
$$n^2+3 = (2k-2)^2+3$$
$$= 4k^2-8k+4+3$$
$$= 4k^2-8k+6+1$$
$$= 2(2k^2-4k+3)+1.$$

So $n^2+3 = 2b+1$ for $b = 2k^2-4k+3 \in \mathbb{Z}$.

Therefore n^3+3 is odd. ∎

8. (12 points) Prove or Disprove: There is a set X for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

TRUE

Proof Let $X = \{\mathbb{N}, 1, 2, 3, 4, 5, \dots\}$

$= \{\{1, 2, 3, 4, 5, \dots\}, 1, 2, 3, 4, 5, \dots\}$

Then $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$. ■

9. (12 points) Prove or Disprove: For all $a, b \in \mathbb{Z}$, if $a \mid b$ and $b \mid a$ then $a = b$.

This is FALSE.

Here is a counterexample:

Let $a = 2$ and $b = -2$.

Then $a|b$ is $2|-2$, which is true.

Also $b|a$ is $-2|2$, which is true.

However, $a \neq b$.