MATH 211 Test #2 November 10, 2016

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**Directions** No calculators. Please put all phones, etc., away.

1. Short Answer:

   (a) Give at least one statement that is logically equivalent to $P \Rightarrow Q$.

   $\sim Q \Rightarrow \sim P$
   $P \wedge \sim Q \Rightarrow (C \wedge \sim C)$
   $\sim P \vee Q$
   } any of these is sufficient

   (b) State DeMorgan's Laws.

   $\sim (P \wedge Q) = \sim P \vee \sim Q$

   $\sim (P \vee Q) = \sim P \wedge \sim Q$

2. Write a truth table to decide if $P \Rightarrow \sim Q$ and $(\sim P) \vee (\sim Q)$ are logically equivalent.

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $P \Rightarrow \sim Q$ | $(\sim P) \vee (\sim Q)$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

Because the columns agree, the two statements $P \Rightarrow \sim Q$ and $(\sim P) \vee (\sim Q)$ are logically equivalent.

3. Suppose the statement $((R \wedge S) \Rightarrow P) \Leftrightarrow \underbrace{(Q \wedge \sim Q)}_{F}$ is **true**. Find the truth values of $R$, $S$ and $P$. (This can be done without a truth table.)

Because $Q \wedge \sim Q$ is FALSE, then $(R \wedge S) \Rightarrow P$ is false also This means $R \wedge S$ is TRUE and $P$ is FALSE. Therefore:

$$\boxed{\begin{array}{l} R = T \\ S = T \\ P = F \end{array}}$$

4. This problem concerns the following statement.
$P$: Given any $x \in \mathbb{R}$, there exists an element $y \in \mathbb{R}$ for which $xy = 1$.

(a) Is the statement $P$ true or false? **Explain.**

This is false because $x = 0 \in \mathbb{R}$, but there is no element $y \in \mathbb{R}$ for which $xy = 0 \cdot y = 1$.

(b) Write the statement $P$ in symbolic form.

$$\boxed{\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1}$$

(c) Form the negation $\sim P$ of your answer from (b), and simplify.

$$\sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1)$$
$$= \exists x \in \mathbb{R}, \sim(\exists y \in \mathbb{R}, xy = 1)$$
$$= \exists x \in \mathbb{R}, \forall y \in \mathbb{R} \sim(xy = 1)$$
$$= \boxed{\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ xy \neq 1}$$

(d) Write the negation $\sim P$ as an English sentence.
(The sentence may use mathematical symbols.)

There is a real number $x$ with the property that $xy \neq 1$ for every real number $y$.

Note: This is true because the number $x = 0 \in \mathbb{R}$ has the property $xy \neq 1$ for every $y \in \mathbb{R}$.

5. A **geometric sequence** with ratio $r$ is a sequence of numbers for which any term is $r$ times the previous term. If the first term of the sequence is $a$, then the sequence is $a, ar, ar^2, ar^3, ar^4, ar^5 \ldots$. Write an algorithm whose input is three numbers $a, r \in \mathbb{R}$, and $n \in \mathbb{N}$, and whose output is the first $n$ terms of the geometric sequence with first term $a$ and ratio $r$.

Algorithm

Input: $a, r \in \mathbb{R}$, $n \in \mathbb{N}$
output: $a, ar, ar^2, ar^3, \ldots ar^{n-1}$

```
begin
   for i := 1 to n do
      output a
      a := a·r
   end
end
```

6. **Prove:** If $a$ is an even integer, then $a^2$ is even. [Direct proof may be easiest.]

Proof (direct) Suppose $a$ is even.
This means $a = 2b$ for some $b \in \mathbb{Z}$.
Then $a^2 = (2b)^2 = 4b^2 = 2(2b^2)$.
So $a^2 = 2k$, where $k = 2b^2 \in \mathbb{Z}$.
Therefore $a^2$ is even. ■

7. **Prove:** If $a$ is an odd integer, then $a^2 + 3a + 5$ is odd. [Direct proof may be easiest.]

Proof (direct)

Suppose $a$ is odd.

Thus $a = 2k+1$ for some $k \in \mathbb{Z}$.

Then $a^2 + 3a + 5 = (2k+1)^2 + 3(2k+1) + 5$

$= 4k^2 + 4k + 1 + 6k + 3 + 5$

$= 4k^2 + 4k + 6k + 9$

$= 4k^2 + 10k + 8 + 1$

$= 2(2k^2 + 5k + 4) + 1.$

The above shows $a^2 + 3a + 5 = 2b + 1$,

where $b = 2k^2 + 5k + 4$.

Therefore $a^2 + 3a + 5$ is odd. ■

8. Suppose $n \in \mathbb{Z}$. **Prove:** If $3 \nmid n^2$, then $3 \nmid n$. [Contrapositive may be easiest.]

Proof (Contrapositive)

Suppose $3 \mid n$.

This means $n = 3a$, where $a \in \mathbb{Z}$.

Then $n^2 = (3a)^2 = 9a^2 = 3 \cdot (3a^2)$

Thus $n^2 = 3b$ where $b = 3a^2 \in \mathbb{Z}$

This means $3 \mid n^2$. ■

9. **Prove:** If $n \in \mathbb{Z}$, then $4 \nmid (n^2+2)$. [Contradiction may be easiest.]

For the sake of contradiction, suppose $n \in \mathbb{Z}$ and $4 \mid (n^2+2)$.
This means $\boxed{n^2+2 = 4b}$ for some $b \in \mathbb{Z}$.
Let's consider two cases.

CASE I Suppose $n$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$.
Then $n^2+2 = 4b$ becomes $(2k)^2+2 = 4b$, which is
$4k^2+2 = 4b$. Then $2 = 4b-4k^2$. Factoring, $2 = 4(b-k^2)$.
Dividing by 2 we get $1 = 2(b-k^2)$ which means that
1 is even, a contradiction

CASE II Suppose $n$ is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$.
Then:

$$\begin{aligned} n^2+2 &= 4b \\ (2k+1)^2+2 &= 4b \\ 4k^2+4k+1+2 &= 4b \\ 1 &= 4b-2-4k-4k^2 \\ 1 &= 2(2b-2-2k-2k^2) \end{aligned}$$

Therefore 1 is even, which is a contradiction ▨

10. Suppose $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. **Prove:** If $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$, then $c \equiv b \pmod{n}$.

Proof (Direct) Suppose $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$
This means $n \mid (a-b)$ and $n \mid (a-c)$.
In turn, we get $\boxed{a-b = nk}$ and $\boxed{a-c = n\ell}$ for $k, \ell \in \mathbb{Z}$.
Subtracting one equation from the other,

$$\begin{aligned} a-b &= nk \\ -a+c &= -n\ell \\ \hline c-b &= nk-n\ell \end{aligned}$$

Therefore $c-b = n(k-\ell)$.
This means $n \mid (c-b)$, and

consequently $c \equiv b \pmod{n}$ ▨