MATH 211 Test #1 ♡ October 5, 2017

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Directions Except for those problems designated **short answer**, you must show and explain your work to get full credit. This test is closed-book and closed-notes. No calculators or other electronic devices. All you will need is something to write with. I will provide scratch paper.

1. (6 points) **Short answer.**

   (a) Write the set $\{2n : n^2 \leq 16\}$ by listing its elements between braces.

   $\boxed{\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}}$

   (b) Write $\{\ldots -2, 3, 8, 13, 18, 28, 33, \ldots\}$ in set-builder notation.

   $\boxed{\{5n+3 : n \in \mathbb{Z}\}}$

   (c) Write the set $(\{1,3\} \times \mathbb{Z}) \cap (\mathbb{Z} \times \{5,6\})$ by listing its elements between braces.

   $\boxed{\{(1,5), (1,6), (3,5), (3,6)\}}$

2. (6 points) **Short answer.** Suppose A and B are sets for which $|A| = m$ and $|B| = n$. Find the following cardinalities.

   (a) $|A \times B| = \boxed{mn}$

   (b) $|\mathscr{P}(A \times B)| = \boxed{2^{mn}}$

   (c) $|\{X \in \mathscr{P}(A) : |X| = 5\}| = \boxed{\binom{m}{5}}$ (This is the number of subsets of A That have cardinality 5)

3. (8 points) **Short answer.** Suppose $A = \{1,3,4,6,9\}$ and $B = \{4,5,6,8,9\}$ are two sets in a universal set $U = \{1,2,3,4,5,6,7,8,9\}$

   (a) $A \cap B = \boxed{\{4, 6, 9\}}$

   (b) $A - \overline{B} = \{1,3,4,6,9\} - \{1,2,3,7\} = \boxed{\{4,6,9\}}$

   (c) $\overline{\emptyset} = \{1,2,3,4,5,6,7,8,9\} = \boxed{U}$

   (d) $(A-B)^2 = \{1,3\}^2 = \{1,3\} \times \{1,3\}$

   $= \boxed{\{(1,1), (1,3), (3,1), (3,3)\}}$

4. (20 points) This question concerns **length-5** lists made from the letters A, B, C, D, E, F, G, H, I.

(a) How many such lists have no repetition and end with a vowel?

8 7 6 5 3

fill this in first with choice of A, E, I

Ans. $8 \cdot 7 \cdot 6 \cdot 5 \cdot 3 = \boxed{5040}$

(b) How many such lists are there if repetition is allowed, and the list contains at least one A?

$\underbrace{9\ 9\ 9\ 9\ 9}_{\text{repetition OK any letter}} - \underbrace{8\ 8\ 8\ 8\ 8}_{\text{repetition OK, any letter but A}}$

By subtraction principle the answer is

$9^5 - 8^5 = \boxed{2681}$

(c) How many such lists are there if repetition is allowed, and the list contains at *exactly* one A?

There are 5 types of lists

A 8 8 8 8 | 8 A 8 8 8 | 8 8 A 8 8 | 8 8 8 A 8 | 8 8 8 8 A

By addition principle the answer is

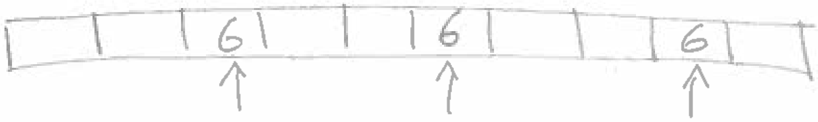
$5 \cdot 8^4 = \boxed{20480}$

(d) How many such lists are there that have no repetition and are in alphabetical order?

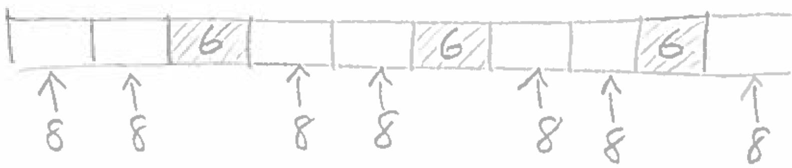
Answer $\binom{9}{5} = \boxed{126}$ because

to make such a list you just choose 5 of 9 letters A, B, C, D, E, F, G, H, I and arrange them in alphabetical order.

5. (10 points) How many 10-digit integers contain no 0's and exactly three 6's?



First choose 3 out of 10 positions for the three 6's There are $\binom{10}{3}$ ways to do This.



Then fill remaining seven positions with a choice of 8 digits per position (any digit but 0 and 6).

Thus the total number of such integers is $\boxed{\binom{10}{3} 8^7}$

6. (10 points) In how many ways can you distribute 30 identical pieces of candy among 4 children?

Children: {Alan, Bill, Clyde, Donna}

Each distribution corresponds to a list

$$\overbrace{* * * \cdots *}^{\text{Alan}} \,|\, \overbrace{* * * \cdots *}^{\text{Bill}} \,|\, \overbrace{* * * \cdots *}^{\text{Clyde}} \,|\, \overbrace{* * * *}^{\text{Donna}}$$

List has length 33, with 30 stars and 3 bars. To make such a list choose 3 out of 33 positions for bars and fill the rest with stars.

Answer: $\binom{33}{3} = \dfrac{33!}{3!\,30!} = \dfrac{33 \cdot 32 \cdot 31}{3 \cdot 2} = \boxed{5456}$

7. (10 points)

(a) Here are the first several rows of Pascal's triangle. Write the next row.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

(b) Use part (a) to find the coefficient of x^3y^3 in $(2x-y)^6$.
Please give the exact (i.e., worked out) value.

Relavant term is $\binom{6}{3}(2x)^3(-y)^3$

$= 20\cdot 2^3x^3(-1)^3y^3 = -20\cdot 8\,x^3y^3$

$= -160x^3y^3$. Thus coefficient is $\boxed{-160}$

8. (10 points) Use the binomial theorem to show that
$3^n = 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + 2^4\binom{n}{4} + \cdots + 2^n\binom{n}{n}$.

By binomial theorem,

$3^n = (1+2)^n$

$= \binom{n}{0}1^n2^0 + \binom{n}{1}1^{n-1}2^1 + \binom{n}{2}1^{n-2}2^2 + \cdots + \binom{n}{n}1^0 2^n$

$= 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n}$

9. (10 points) A department consists of 5 men and 7 women. From this department you select a committee with 3 men and 2 women. In how many ways can you do this?

$$\underbrace{\binom{5}{3}}_{\text{Choose 3 men}}\underbrace{\binom{7}{2}}_{\text{Choose 2 women}} = \frac{5!}{3!\,2!}\,\frac{7!}{2!\,5!} = \frac{7!}{3!\,2!\,2!}$$

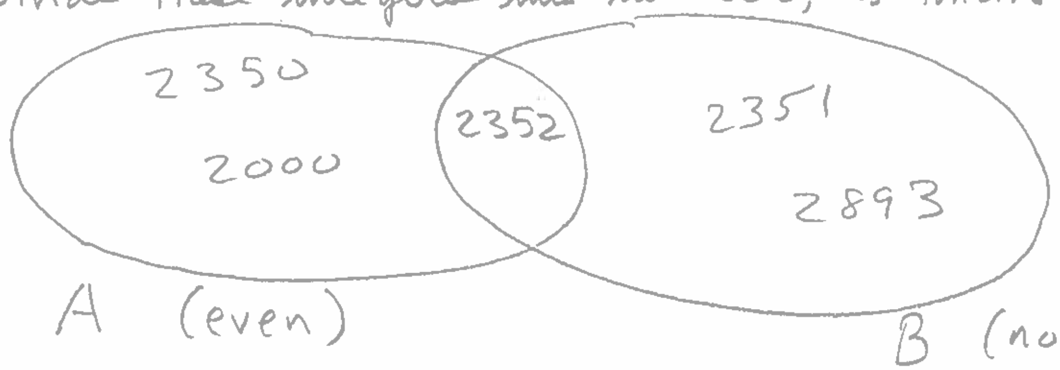
$$= \frac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\,2!\,2!}$$

$$= 7\cdot 3\cdot 5\cdot 2$$

$$= \boxed{210 \text{ ways}}$$

10. (10 points) How many 4-digit positive integers are there that are even or contain no 0's?

Divide these integers into two sets, as follows:



Numbers in A: $\square\square\square\square$ with $9\cdot 10\cdot 10\cdot 5$ choices

Thus $|A| = 9\cdot 10\cdot 10\cdot 5 = 4500$

Numbers in B: $\square\square\square\square$ with $9\ 9\ 9\ 9$ choices

Thus $|B| = 9^4 = 6561$

Also, a number in $A\cap B$ looks like $\square\square\square\square$ with $9\ 9\ 9\ 4$ choices

Thus $|A\cap B| = 9^3\cdot 4 = 2916$.

By the inclusion-exclusion principle, our answer is $|A\cup B| = |A| + |B| - |A\cap B| = 4500 + 6561 - 2916 = \boxed{8145}$